Writing down all the units for each term:

\[
[ML^{-3}] \left[ \frac{LT^{-1}}{T} \right] + [LT^{-1}] \left( \frac{LT^{-1}}{L} \right) + [LT^{-1}] \left( \frac{LT^{-1}}{L} \right) + [LT^{-1}] \left( \frac{LT^{-1}}{L} \right) =
\]

\[
\left[ \frac{ML^{-1}T^{-2}}{L} \right] + [ML^{-3}] + [ML^{-1}T^{-1}] \left[ \left( \frac{LT^{-1}}{L^2} \right) + \left( \frac{LT^{-1}}{L^2} \right) + \left( \frac{LT^{-1}}{L^2} \right) \right]
\]

And simplifying:

\[
\left[ \frac{ML^{-2}T^{-2}}{L^2} \right] + \left[ ML^{-2}T^{-2} \right] + \left[ ML^{-2}T^{-2} \right] + \left[ ML^{-2}T^{-2} \right] =
\]

\[
\left[ ML^{-2}T^{-2} \right] + \left[ ML^{-2}T^{-2} \right] + \left[ \left( ML^{-2}T^{-2} \right) + \left( ML^{-2}T^{-2} \right) + \left( ML^{-2}T^{-2} \right) \right]
\]

Hence the equation is consistent. This is the Navier-Stokes equation and is basically \( F = ma \) in a fluid (note the units are in Force/Volume, e.g., \( [ML^{-2}T^{-2}] = \left[ \frac{MLT^{-2}}{L^3} \right] \) as in fluids we tend to work with units that are normalized by volume). We will see this equation again later in the course!
1.10 The Stokes-Oseen formula [10] for drag on a sphere at low velocity \( V \) is:

\[
F = 3\pi\mu DV + \frac{9\pi}{16} \rho V^2 D^2
\]

where \( D \) = sphere diameter, \( \mu \) = viscosity, and \( \rho \) = density. Is the formula homogeneous?

**Solution:** Write this formula in dimensional form, using Table 1-2:

\[
\{F\} = \{3\pi\} \{\mu\} \{D\} \{V\} + \left[\frac{9\pi}{16}\right] \{\rho\} \{V\}^2 \{D\}\ ?
\]

or:

\[
\left[\frac{ML}{T^2}\right] = \{1\} \left[\frac{M}{LT}\right] \{L\} \left[\frac{L}{T}\right] + \{1\} \left[\frac{M}{L^3}\right] \left[\frac{L^2}{T^2}\right] \{L^2\} \ ?
\]

where, hoping for homogeneity, we have assumed that all constants \((3, \pi, 9, 16)\) are *pure*, i.e., \{unity\}. Well, yes indeed, all terms have dimensions \{ML/T^2\}! Therefore the Stokes-Oseen formula (derived in fact from a theory) is **dimensionally homogeneous**.

---

**P1.11** In English Engineering units, the specific heat \( c_p \) of air at room temperature is approximately 0.24 Btu/(lbm·°F). When working with kinetic energy relations, it is more appropriate to express \( c_p \) as a velocity-squared per absolute degree. Give the numerical value, in this form, of \( c_p \) for air in (a) SI units, and (b) BG units.

**Solution:** From Appendix C, *Conversion Factors*, 1 Btu = 1055.056 J (or N·m) = 778.17 ft-lbf, and 1 lbm = 0.4536 kg = (1/32.174) slug. Thus the conversions are:

**SI units:**

\[
\frac{0.24 \text{ Btu}}{\text{lbm °F}} = 0.24 \frac{1055.056 \text{ N·m}}{(0.4536 \text{ kg})(1\text{K} / 1.8)} = 1005 \frac{\text{N·m}}{\text{kg·K}} = 1005 \frac{m^2}{s^2 \text{K}} \quad \text{Ans.}(a)
\]

**BG units:**

\[
\frac{0.24 \text{ Btu}}{\text{lbm °F}} = 0.24 \frac{778.17 \text{ ft·lbf}}{[(1/32.174)\text{slug})(1°\text{R})} = 6009 \frac{\text{ft·lbf}}{\text{slug·°R}} = 6009 \frac{\text{ft}^2}{s^2 \text{°R}} \quad \text{Ans.}(b)
\]

---

1.12 For low-speed (laminar) flow in a tube of radius \( r_o \), the velocity \( u \) takes the form:

\[
u = B \frac{\Delta p}{\mu} \left( r_o^2 - r^2 \right)
\]

where \( \mu \) is viscosity and \( \Delta p \) the pressure drop. What are the dimensions of \( B \)?
**Solution:** Write out the dimensions of each of the terms in the formula:

\[ \{h_f\} = \{L\} ; \{L_o\} = \{L\} ; \{Q\} = \{L^3/T\} ; \{A\} = \{L^2\} ; \{C_h\} = \{1\} ; \{D\} = \{L\} \]

Use these dimensions in the equation to determine \{0.551\}. Since \(h_f\) and \(L_o\) have the same dimensions \{L\}, it follows that the quantity in parentheses must be dimensionless:

\[
\left\{ \frac{Q}{0.551AC_hD^{0.63}} \right\} = \{1\} = \left\{ \frac{L^3/T}{\{L\}^{\{1\}} \{L\}^{0.63}} \right\} = \left\{ \frac{L^{0.37}}{(0.551)T} \right\} = \{1\}
\]

It follows that \{0.551\} = \{L^{0.37}/T\} \quad \text{Ans.}

The constant has dimensions; therefore beware. The formula is valid only for water flow at high (turbulent) velocities. The density and viscosity of water are hidden in the constant 0.551, and the wall roughness is hidden (approximately) in the numerical value of \(C_h\).

---

**1.16** Test the dimensional homogeneity of the boundary-layer x-momentum equation:

\[
\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \rho g_x + \frac{\partial \tau}{\partial y}
\]

**Solution:** This equation, like all theoretical partial differential equations in mechanics, is dimensionally homogeneous. Test each term in sequence:

\[
\left\{ \rho u \frac{\partial u}{\partial x} \right\} = \left\{ \rho v \frac{\partial u}{\partial y} \right\} = \left\{ \frac{M \cdot L}{L^3 \cdot T} \right\} = \left\{ \frac{M}{L^2 \cdot T^2} \right\}; \quad \left\{ \frac{\partial p}{\partial x} \right\} = \left\{ \frac{M}{L} \right\} = \left\{ \frac{M}{L^2 \cdot T^2} \right\}
\]

\[
\left\{ \rho g_x \right\} = \left\{ \frac{M}{L^3 \cdot T^2} \right\}; \quad \left\{ \frac{\partial \tau}{\partial x} \right\} = \left\{ \frac{M}{L} \right\} = \left\{ \frac{M}{L^2 \cdot T^2} \right\}
\]

All terms have dimension \{ML^{-2}T^{-2}\}. This equation may use any consistent units.

---

**1.17** Investigate the consistency of the Hazen-Williams formula from hydraulics:

\[
Q = 61.9D^{2.63} \left( \frac{\Delta p}{L} \right)^{0.54}
\]

What are the dimensions of the constant “61.9”? Can this equation be used with confidence for a variety of liquids and gases?
Solution: Write out the dimensions of each side of the equation:

\[
\{Q\} = \left\{ \frac{L^3}{T} \right\} = \{61.9\} \{D^{2.63}\} \left\{ \frac{\Delta p}{L} \right\}^{0.54} = \{61.9\} \{L^{2.63}\} \left\{ \frac{M/LT^2}{L} \right\}^{0.54}
\]

The constant 61.9 has fractional dimensions: \(\{61.9\} = \{L^{1.45} T^{-0.08} M^{-0.54}\}\) Ans.

Clearly, the formula is extremely inconsistent and cannot be used with confidence for any given fluid or condition or units. Actually, the Hazen-Williams formula, still in common use in the watersupply industry, is valid only for water flow in smooth pipes larger than 2-in. diameter and turbulent velocities less than 10 ft/s and (certain) English units. This formula should be held at arm’s length and given a vote of “No Confidence.”

*1.18 (“*” means “difficult”—not just a plug-and-chug, that is) For small particles at low velocities, the first (linear) term in Stokes’ drag law, Prob. 1.10, is dominant, hence \(F = KV\), where \(K\) is a constant. Suppose a particle of mass \(m\) is constrained to move horizontally from the initial position \(x = 0\) with initial velocity \(V = V_0\). Show (a) that its velocity will decrease exponentially with time; and (b) that it will stop after travelling a distance \(x = mV_0/K\).

**Solution:** Set up and solve the differential equation for forces in the \(x\)-direction:

\[
\sum F_x = -\text{Drag} = ma_x, \quad \text{or:} \quad -KV = m\frac{dV}{dt}, \quad \text{integrate} \int_{V_0}^{V} \frac{dV}{V} = -\int_{0}^{t} \frac{m}{K} dt
\]

Solve \(V = V_0 e^{-mt/K}\) and \(x = \int_{0}^{t} V \, dt = \frac{mV_0}{K} \left( 1 - e^{-mt/K} \right)\) Ans. (a,b)

Thus, as asked, \(V\) drops off exponentially with time, and, as \(t \to \infty, x = K \frac{V_0}{m}\)

P1.19 In his study of the circular hydraulic jump formed by a faucet flowing into a sink, Watson [53] proposes a parameter combining volume flow rate \(Q\), density \(\rho\) and
(b) Cooling at constant volume means $\rho$ stays the same and the new temperature is 283K. Thus

$$p_3 = \rho_3 RT_3 = (5.00 \frac{kg}{m^3})(208 \frac{m^2}{s^2}K)(283K) = 294,000 Pa = 294 kPa \quad \text{Ans.}(b)$$

1.32 A blimp is approximated by a prolate spheroid 90 m long and 30 m in diameter. Estimate the weight of 20°C gas within the blimp for (a) helium at 1.1 atm; and (b) air at 1.0 atm. What might the difference between these two values represent (Chap. 2)?

**Solution:** Find a handbook. The volume of a prolate spheroid is, for our data,

$$V = \frac{2}{3} \pi LR^2 = \frac{2}{3} \pi (90 m)(15 m)^2 \approx 42412 \text{ m}^3$$

Estimate, from the ideal-gas law, the respective densities of helium and air:

(a) $\rho_{\text{helium}} = \frac{p_{\text{He}}}{R_{\text{He}}T} = \frac{1.1(101350)}{2077(293)} \approx 0.1832 \frac{\text{kg}}{\text{m}^3}$;

(b) $\rho_{\text{air}} = \frac{p_{\text{air}}}{R_{\text{air}}T} = \frac{101350}{287(293)} \approx 1.205 \frac{\text{kg}}{\text{m}^3}$.

Then the respective gas weights are

$$W_{\text{He}} = \rho_{\text{He}}gV = \left(0.1832 \frac{\text{kg}}{\text{m}^3}\right)\left(9.81 \frac{\text{m}}{\text{s}^2}\right)(42412 \text{ m}^3) \approx 76000 \text{ N} \quad \text{Ans.} (a)$$

$$W_{\text{air}} = \rho_{\text{air}}gV = (1.205)(9.81)(42412) \approx 501000 \text{ N} \quad \text{Ans.} (b)$$

The difference between these two, 425000 N, is the *buoyancy*, or lifting ability, of the blimp. [See Section 2.8 for the principles of buoyancy.]

P1.33 Experimental data [55] for the density of n-pentane liquid for high pressures, at 50°C, are listed as follows:

<table>
<thead>
<tr>
<th>Pressure, MPa</th>
<th>0.01</th>
<th>10.23</th>
<th>20.70</th>
<th>34.31</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density, kg/m³</td>
<td>586.3</td>
<td>604.1</td>
<td>617.8</td>
<td>632.8</td>
</tr>
</tbody>
</table>
The formula is of course valid only for laminar (nonturbulent) steady viscous flow.
(b) Since the center plate separates the two fluids, they may have separate, unrelated shear stresses, and there is no necessary relation between the two viscosities.

1.49 An amazing number of commercial and laboratory devices have been developed to measure fluid viscosity, as described in Ref. 27. Consider a concentric shaft, fixed axially and rotated inside the sleeve. Let the inner and outer cylinders have radii \( r_i \) and \( r_o \), respectively, with total sleeve length \( L \). Let the rotational rate be \( \Omega \) (rad/s) and the applied torque be \( M \). Using these parameters, derive a theoretical relation for the viscosity \( \mu \) of the fluid between the cylinders.

**Solution:** Assuming a linear velocity distribution in the annular clearance, the shear stress is
\[
\tau = \mu \frac{\Delta V}{\Delta r} \approx \mu \frac{\Omega r_i}{r_o - r_i}
\]
This stress causes a force \( dF = \tau \, dA = \tau (r_i \, d\theta)L \) on each element of surface area of the inner shaft. The moment of this force about the shaft axis is \( dM = r_i \, dF \). Put all this together:
\[
M = \int r_i \, dF = \int_0^{2\pi} r_i \mu \frac{\Omega r_i}{r_o - r_i} L \, d\theta = \frac{2\pi \mu \Omega r_i^3 L}{r_o - r_i}
\]
Solve for the viscosity:
\[
\mu \approx M(r_o - r_i) / \left\{ 2\pi \Omega r_i^3 L \right\}
\]
Ans.

1.50 A simple viscometer measures the time \( t \) for a solid sphere to fall a distance \( L \) through a test fluid of density \( \rho \). The fluid viscosity \( \mu \) is then given by
\[
\mu \approx \frac{W_{\text{net}} t}{3\pi D L} \quad \text{if} \quad t \geq \frac{2\rho DL}{\mu}
\]
where \( D \) is the sphere diameter and \( W_{\text{net}} \) is the sphere net weight in the fluid.
(a) Show that both of these formulas are dimensionally homogeneous. (b) Suppose that a 2.5 mm diameter aluminum sphere (density 2700 kg/m\(^3\)) falls in an oil of density 875 kg/m\(^3\). If the time to fall 50 cm is 32 s, estimate the oil viscosity and verify that the inequality is valid.
P1.55 A block of weight \( W \) is being pulled over a table by another weight \( W_o \), as shown in Fig. P1.55. Find an algebraic formula for the steady velocity \( U \) of the block if it slides on an oil film of thickness \( h \) and viscosity \( \mu \). The block bottom area \( A \) is in contact with the oil. Neglect the cord weight and the pulley friction.

Solution: This problem is a lot easier to solve than to set up and sketch. For steady motion,

\[
\sum F_{x,\text{block}} = 0 = \tau A - W_o = (\mu \frac{U}{h}) A - W_o
\]

Solve for \( U = \frac{W_o h}{\mu A} \) \( \text{Ans.} \)

there is no acceleration, and the falling weight balances the viscous resistance of the oil film:

The block weight \( W \) has no effect on steady horizontal motion except to smush the oil film.

1.56* For the cone-plate viscometer in Fig. P1.56, the angle is very small, and the gap is filled with test liquid \( \mu \). Assuming a linear velocity profile, derive a formula for the viscosity \( \mu \) in terms of the torque \( M \) and cone parameters.

Solution: For any radius \( r \leq R \), the liquid gap is \( h = r \tan \theta \). Then

\[
d(\text{Torque}) = dM = \tau dA_w r = \left( \mu \frac{\Omega r}{r \tan \theta} \right) \left( 2\pi r \frac{dr}{\cos \theta} \right) r, \quad \text{or}
\]

\[
M = \frac{2\pi \Omega \mu}{\sin \theta} \int_0^R r^2 dr = \frac{2\pi \Omega \mu R^3}{3 \sin \theta}, \quad \text{or:} \quad \mu = \frac{3M \sin \theta}{2\pi R^3} \quad \text{Ans.}
\]
P1.57 Extend the steady flow between a fixed lower plate and a moving upper plate, from Fig. 1.8, to the case of two immiscible liquids between the plates, as in Fig. P1.57.

(a) Sketch the expected no-slip velocity distribution \( u(y) \) between the plates. (b) Find an analytic expression for the velocity \( U \) at the interface between the two liquid layers. (c) What is the result if the viscosities and layer thicknesses are equal?

Solution: We begin with the hint, from Fig. 1.8, that the shear stress is constant between the two plates. The velocity profile would be a straight line in each layer, with different slopes:

Here we have drawn the case where \( \mu_2 > \mu_1 \), hence the upper profile slope is less. (b) Set the two shear stresses equal, assuming no-slip at each wall:

\[
\tau_1 = \mu_1 \frac{(U - 0)}{h_1} = \tau_2 = \mu_2 \frac{(V - U)}{h_2}
\]

Solve for \( U = V \left[ \frac{1}{1 + \frac{\mu_1 h_2}{\mu_2 h_1}} \right] \)  

Ans. (b)