1 (1.14) Figure P1.14 shows the flow of water over a dam. The volume flow $Q$ is known to depend only on crest width $B$, acceleration of gravity $g$, and upstream water height $H$ above the dam crest. It is further known that $Q$ is proportional to $B$.

What is the form of the only possible dimensionally homogeneous relation for this flow rate?

Solution: This is a preview of how we will solve problems using dimensional analysis.

We know that $Q = f(B, H, g)$

and $Q \propto B$

$\Rightarrow$ $Q = B \cdot f(H, g)$

Let's look at the dimensions:

$\left[ \frac{L^2}{T} \right] = \left[ L \right] \left[ f(H, g) \right]$  

For consistency, $f(H, g)$ must have dimensions of $\frac{L^3}{T}$, how do we get it?

$[H] = L$  
$[g] = LT^{-2}$

only $g$ has dimensions of $T$

$\Rightarrow$ use $[\sqrt{g}] = \frac{L^{3/2}}{T}$

Now for $H$:

$\left[ H^n \sqrt{g} \right] = L^n \frac{L^{3/2}}{T} = \frac{L^3}{T}  

\Rightarrow n = \frac{3}{2}$

$Q \propto B \sqrt[3/2]{H} \sqrt{g}$

$Q = C \cdot B \cdot H^{3/2} \cdot g^{1/2}$

($^*$dimensionless constant.)

2 (1.41) An aluminum cylinder weighing 30 N, 6 cm in diameter and 40 cm long, is falling concentrically through a long vertical tube of diameter 6.04 cm. The clearance is filled with SAE 50 oil at 80°C. Estimate the terminal fall velocity.

Neglect air drag and assume a linear velocity distribution in the oil.

At terminal velocity:

$W = F_T$

$= 2 \pi \text{wall area}$

$= \left( \mu \cdot \frac{dV}{dz} \right) \cdot \text{TDL}$

With a linear velocity profile:

$W = \mu \cdot \frac{V}{6} \cdot \text{TDL}$

$V = \frac{W}{\mu \cdot \text{TDL}} = \frac{(30 \text{ N}) (2 \times 10^{-4} \text{ m})}{(0.86 \frac{\text{ kg}}{\text{ m s}^2})(6 \times 10^{-2} \text{ m})(0.4 \text{ m})}$

$V = 9.25 \times 10^{-2} \text{ m/s}$
3. (1.57) Extend the steady flow between a fixed lower plate and a moving upper plate, to the case of two immiscible liquids between the plates, as in the figure:

\[ \begin{align*}
\text{Solution:} \\
&\text{a) Sketch the expected non-slip velocity distribution } u(y) \text{ between plates.} \\
&\text{b) Find an analytic expression for the velocity } U \text{ at the interface between the two liquid layers.} \\
&\text{c) What is the result of (b) if the viscosities and layer thickness are equal?}
\end{align*} \]

\[ \begin{align*}
&u = \frac{V}{h_2} \quad \text{(if } M_2 > M_1) \\
&\frac{\partial u}{\partial y} = \frac{V}{h_1} \\
&\text{if } h_1 = h_2, \quad U = \frac{1}{2} V
\end{align*} \]

4. Viscous clock.
You are constructing a timer which is to mimic the behavior of the second hand on a clock. That is, you want to design a system where the rotation rate of the system is one revolution per minute.

You have built the system below:

\[ \begin{align*}
&\text{which utilizes glycerin in the gap between the annulus (inner } R_3 \text{ and outer } R_2) \text{ and a fixed plate to control the rotation rate. You hang an old fishing weight that has a mass } 0.813 \text{ on a string that you have wound around the wheel with radius } R_1. \text{ If the relevant dimensions are } R_1 = 30 \text{ cm, } R_2 = 10 \text{ cm, } R_3 = 40 \text{ cm, what is the distance } h \text{ you need to set the gap at in order to obtain the desired rotation rate?}
\end{align*} \]

Torque from the fishing weight: \[ T_0 = MgR_1 \] 

Viscous torque:

\[ \tau = \int r \, dA = \int 2\pi r \, dA = \int 2\pi r \, dA \]

\[ \tau = \int 2\pi r \, dA = M \frac{\omega}{r} \]

\[ \frac{2\pi}{T} = \text{revolution period} \]

\[ dA = 2\pi r \, dr \]

\[ \tau = \int 2\pi r \, dA = \int 2\pi r \, dA \]

\[ M = \frac{2\pi \omega}{h} \]

\[ M = \frac{2\pi \omega}{h} \]

\[ h = \frac{2\pi \omega}{M} \]

\[ h = \frac{2\pi \omega}{M} \]

\[ h = \frac{2\pi \omega}{M} \]

\[ h = 1.0 \times 10^{-4} \text{ m} \]
(5.64) Determine the maximum diameter (in mm) of a solid aluminum ball, density \( \rho = 2700 \text{ kg/m}^3 \), which will float on a clean water-air surface at 20°C.

At the maximum weight, \( \theta = 0 \)

Weight (W) must be balanced by the surface tension force.

\[
T = \frac{8 \pi D^3}{6} \sqrt{\eta \rho_g}
\]

\[
D = \sqrt[3]{\frac{6 T}{\pi \eta \rho_g}}
\]

\[
D = \left( \frac{6 \times 0.0728 \text{ N/m}}{(2700 \text{ kg/m}^3)(9.81 \text{ m/s}^2)} \right)^{1/3}
\]

\[
D = 4.06 \text{ mm}
\]

6 Consider the water strider from lesson 3. Their feet are coated in wax, and well approx. as being cylindrical. If careful inspection reveals that:

a) the contact line strikes their foot at the point of horizontal diameter

b) the contact angle of the wax-foot-water system is 120°, and

c) each foot (6 of them) is 7 mm long, with a diameter of 0.2 mm

Why is the maximum weight that a water strider can be to walk on a fresh water body?

Sketch the cross section of the foot and show the forces involved.

\[
\frac{L}{D} = \frac{7 \text{ mm}}{0.2 \text{ mm}} = 35 \quad L \gg D
\]

Ignore contact line at both ends.

\[
F_T = 2LT \cos(180° - \theta) = \frac{W}{6}
\]

\[
W = 12(LT \cos(180° - \theta))
\]

\[
W = 12 \left(7 \times 10^{-3} \text{ m}\right) \left(9.8 \times 10^{-2} \text{ N/m}\right) \cos(180° - 12°)
\]

\[
W = 3 \times 10^{-3} \text{ N}
\]

Is buoyancy important?

\[
F_B = \text{Weight of displaced fluid}
\]

\[
= 6 \chi_{w} \left( \frac{\pi D^4}{6} \right) = 6 \left(10000 \text{ N/m}^3\right) \frac{\pi}{6} (2 \times 10^{-3} \text{ m})^4
\]

\[
F_B = 7 \times 10^{-6} \text{ N} < < 3 \times 10^{-3} \text{ N}
\]

\(
\Rightarrow \text{Neglecting buoyancy is OK.}
\)
A 2-D unsteady velocity field is given by:
\[ u = x(1 + 2t) \]
\[ v = y \]
Find the equation of the time-varying streamlines that all pass through the point \((x_0, y_0)\) at some time \(t\).
Sketch a few of these.

Solution:
Recall \[ \frac{dy}{u} = \frac{dx}{v} \]
\[ \frac{dx}{x(1 + 2t)} = \frac{dy}{y} \]
New integrate:
\[ \int\frac{1}{x(1 + 2t)} \, dx = \int \frac{dy}{y} \]
\[ \frac{1}{1 + 2t} \ln x = \ln y + C \]
\[ \ln x = \ln y + C(1 + 2t) \]
\[ \ln x^\frac{n}{2} = \ln y + C \]
\[ C = \frac{n}{2} \ln x^0 - \ln y_0 \]
\[ y = y_0 \exp \left( ^\frac{n}{2} x_0 \right) \]
(2) into (1)
\[ y = y_0 \exp \left( ^\frac{n}{2} x_0 \right) \]
\[ \frac{y}{y_0} = \exp \left( ^\frac{n}{2} \frac{x}{x_0} \right) \]
Plot \[ y/y_0 \] for some values of \(t\).

8

(i) Rank the vessels in increasing order of bottom pressure.
\[ p_a = \gamma h \Rightarrow p_a = p_a = p_a = p_a = p_a = p_a \]
**Same bottom pressure**

(ii) Assume all vessels are shown in cross section and that cross section is constant into the page, then the bottoms of the vessels are all rectangles.
Which case does the pressure at the bottom times the area of the rectangle = weight of the fluid on the container?
In which case \[ pA = \gamma h \] ? (a)
For all cases, \[ p = \gamma h \] (a)
\[ pA - \gamma hA = \gamma A \] (b)
In which case \[ hA = \gamma A \] ? Only in vessel 2.

(iii) For any cases that didn't make your list in part (ii), explain what is going on using a sketch that shows where forces are being generated.

PA < \( \gamma A \)
Vertical component of pressure forces on the sides make up for differences in \( pA - \gamma A \).

PA > \( \gamma A \)
Vertical component of pressure forces on the sides make up for the difference \( pA - \gamma A \).

PA > \( \gamma A \)
Vertical pressure forces on upper wall balance the difference.
In the figure, pressure gauge A reads 1.61 kPa (gage). The tanks are at 20°C. Determine the elevations $z$, in meters, of the liquid levels in the open manometer tubes B and C.

\[ \gamma_{\text{air}} = 12.0 \text{ kN/m}^3 \]
\[ \gamma_{\text{gasoline}} = 8670 \text{ kN/m}^3 \]
\[ \gamma_{\text{glycerin}} = 12360 \text{ kN/m}^3 \]

**Solution:**

From point A to B:

\[ P_A + \gamma_{\text{air}} h_A + \gamma_{\text{gasoline}} (h_{\text{gasoline}} - h_A) = \gamma_{\text{gasoline}} (z_B - h_A - h_{\text{glycerin}}) = P_B = 0 \]

\[ z_B = \frac{P_A + \gamma_{\text{air}} h_A + \gamma_{\text{gasoline}} h_{\text{gasoline}} + \gamma_{\text{glycerin}} h_{\text{glycerin}}}{\gamma_{\text{gasoline}}} \]

\[ = \frac{1500 \text{ N/m}^3 + (12 \text{ N/m}^3)(2m) + 1.5m + 1.0m}{6670 \text{ kN/m}^3} \]

\[ z_B = 2.73 \text{ m} \]

Likewise, from point A to C:

\[ P_A + \gamma_{\text{air}} h_A + \gamma_{\text{gasoline}} h_{\text{gasoline}} + \gamma_{\text{glycerin}} (h_{\text{glycerin}} - h_A) = \gamma_{\text{glycerin}} (z_C - h_A - h_{\text{gasoline}}) = P_C = 0 \]

\[ z_C = \frac{P_A + \gamma_{\text{air}} h_A + \gamma_{\text{gasoline}} h_{\text{gasoline}} + \gamma_{\text{glycerin}} h_{\text{glycerin}}}{\gamma_{\text{glycerin}}} \]

\[ = \frac{1500 \text{ N/m}^3 + (12 \text{ N/m}^3)(2m) + (6670 \text{ kN/m}^3)(1.5m) + 1.0m}{12360 \text{ kN/m}^3} \]

\[ z_C = 1.93 \text{ m} \]
(2.14) For the three-liquid system shown, compute $h_1$ and $h_2$. Neglect air density.

![Diagram of three-liquid system]

- Water: $h_A = 2.7 \text{ cm}$
- Mercury: $h_B = h_C = 5 \text{ cm}$

Use gage pressure $\Rightarrow P_A = P_B = P_C = 0$

From A to B:

\[ \gamma_{\text{water}} h_A + \gamma_{\text{mercury}} h_B - \gamma_{\text{mercury}} h_A = 0 \]

\[ h_1 = \frac{\gamma_{\text{mercury}} h_B - \gamma_{\text{water}} h_A}{\gamma_{\text{mercury}}} \]

\[ = h_B - \frac{\gamma_{\text{water}} h_A}{\gamma_{\text{mercury}}} \]

\[ = 0.089 \text{ m} \cdot \left( \frac{9790 \text{ N/m}^3}{133100 \text{ N/m}^3} \right) \cdot (0.27 \text{ m}) \]

\[ h_1 = 0.06 \text{ m} \]

From B to C:

\[ \gamma_{\text{mercury}} h_B - \gamma_{\text{mercury}} h_C - \gamma_{\text{oil}} h_2 = 0 \]

\[ h_2 = \frac{\gamma_{\text{mercury}} (h_B - h_C)}{\gamma_{\text{oil}}} \]

\[ = \frac{133100 \text{ N/m}^3}{7636 \text{ N/m}^3} (0.089 \text{ m} - 0.058 \text{ m}) \]

\[ h_2 = 0.52 \text{ m} \]

(2.55) Gate AB is 5 ft wide into the paper, hinged at A, and restrained by a stop at B. The water is at 20°C. Compute:

a) the force on stop B, and
b) the reactions at A,

c) the water depth $h = 9.5 \text{ ft}$

![Diagram of gate AB]

\[ \gamma = 62.4 \text{ lb/ft}^3 \]

\[ b = 5 \text{ ft} \]

\[ h = 9.5 \text{ ft} \]

Solution:

Find $F_a$ on the gate:

\[ F_a = \frac{\gamma h c}{12} \]

\[ F_a = \frac{(62.4 \text{ lb/ft}^3) (9.5 \text{ ft})}{12} \]

\[ F_a = 426 \text{ lb} \]

And also $F_a$:

\[ F_a = \frac{\gamma h c}{12} \]

\[ = \frac{(62.4 \text{ lb/ft}^3) (9.5 \text{ ft})}{12} \]

\[ F_a = 426 \text{ lb} \]

The forces acting on the gate AB are:

\[ F_a = 426 \text{ lb} \]

(4) $z_{MA} = F_a B_a - B_x L = 0$

\[ B_x = F_a \frac{y}{z} = 9360 \text{ lb} \left( \frac{2.5 \text{ ft}}{1 \text{ ft}} \right) = 54,000 \text{ lb} \]

\[ z_{Fe} = F_a - A_y - B_x = 0 \]

\[ A_x = F_a - B_x = 9360 \text{ lb} - 54,000 \text{ lb} = 4260 \text{ lb} \]

\[ z_{Fe} = A_z - W = 0 \] (If $W_{\text{gate}}$ is negligible, $A_z = 0$)

\[ B_x = 54,000 \text{ lb} \]

\[ A_x = 4260 \text{ lb} \]

\[ A_z = W_{\text{gate}} \]
(2.70) The swing-check valve covers a 22.86 cm diameter opening in the slanted wall. The hinge is 15 cm from the centerline, as shown. The valve will open when the hinge moment is 50 Nm. Find the value of $h$ for the water to cause this condition.

![Diagram of a swing-check valve and related geometry]

Solution:

What is the moment on the valve?

$F_a = \gamma h A = \gamma h \frac{\pi D^2}{4}$

Where is it applied?

$y_a = \frac{L \gamma \sin \theta'}{F_a}$

$y_a = \frac{\pi D^2}{84} \times \frac{16 h}{y_a}$

$y_R = \frac{D^2 \sin \theta'}{16 h}$

We are neglecting the weight of the valve, thus only $F_a$ will contribute to the moment of 50 Nm required to open the valve.

$\omega M_{h=0} = 50 \text{Nm} - F_a (L+y_R) = 0$

$F_a (L+y_R) = 50 \text{Nm}$

$\gamma h \frac{\pi D^2}{4} (L + \frac{D^2 \sin \theta'}{16 h}) = 50 \text{Nm}$

$h \left( \frac{\pi D^2}{4} \right) + \frac{\pi D^2 \sin \theta'}{64} = 50 \text{Nm}$

$h = \left( \frac{50 \text{Nm} - \frac{\pi D^2 \sin \theta'}{64}}{\frac{\pi D^2}{4}} \right) \left( \frac{4}{118 \pi} \right)$

$h = \frac{(50 \text{Nm})(4)}{(0.15 m)(7794 \text{N/m}^3) \pi (0.2286 m)^2} \left( \frac{0.2286 \text{m}^2 \sin 30}{18 (0.15 \text{m})} \right)$

$h = 0.22 \text{ m}$