Review

- **Buoyancy** – weight of fluid displaced by a body.

- **Stability** - overturning moment or restoring moment from force couple between $\vec{F}_B$ and $\vec{W}$.

- **System** – A particular collection of mass.

- **Control volume** – A particular volume in space.

- **Volume flow rate & Mass Flow Rate**

- **Flux** – the amount of stuff crossing a unit area in a unit of time

- **Reynolds Transport Theorem**

$$\frac{dB_{sys}}{dt} = \frac{dB_{CV}}{dt} + \rho_{out} \beta_{out} Q_{out} - \rho_{in} \beta_{in} Q_{in}$$

### 3.3 Conservation of Mass

We can derive the equation for conservation of mass, also known as the *continuity equation*, by simply letting $B = m$ in the Reynolds Transport Theorem. Therefore, $\beta = 1$ and we have

$$\frac{dm_{sys}}{dt} = \frac{d}{dt} \int_{C.V.} \rho \, d\mathcal{V} + \int_{C.S.} \rho (\vec{v} \cdot \vec{n}) \, dA = 0$$

Now, if we have a fixed control volume then

$$\int_{C.V.} \frac{\partial \rho}{\partial t} \, d\mathcal{V} + \int_{C.S.} \rho (\vec{v} \cdot \vec{n}) \, dA = 0$$

And it the flow is at steady state:

$$\int_{C.S.} \rho (\vec{v} \cdot \vec{n}) \, dA = 0$$

If we have one-dimensional inlets and outlets

$$\sum (\rho_i A_i V_i)_{in} = \sum (\rho_i A_i V_i)_{out}$$

or

$$\sum (\dot{m}_i)_{in} = \sum (\dot{m}_i)_{out}$$
3.3.1 A Note on Average Velocity

The velocity $V$ in $\dot{m} = \rho Q = \rho AV$ represents the spatially averaged velocity across $A$ and hence really $\langle V \rangle$ where

$$\langle V \rangle = \frac{\int_A \rho (\vec{v} \cdot \vec{n}) dA}{\rho A}$$

Only if $\rho$ and $V$ are not functions of $A$ does $\langle V \rangle = V$ (i.e., only if the flow is truly one-dimensional).

Example

$$\frac{dh}{dt} = \frac{\rho_1 A_1 V_1 + \rho_2 A_2 V_2}{\rho w A}$$

3.3.2 Incompressible Flow, Fixed C.V.

For a fixed C.V. we have

$$0 = \int_{C.V.} \frac{\partial \rho}{\partial t} dV + \int_{C.S.} \rho (\vec{v} \cdot \vec{n}) dA$$

If the flow is incompressible then $\frac{\partial \rho}{\partial t} = 0$ and we have

$$0 = \int_{C.S.} \rho (\vec{v} \cdot \vec{n}) dA = \int_{C.S.} (\vec{v} \cdot \vec{n}) dA$$

Which states that whatever flows into the control volume must flow out since there is no change in storage in the control volume (the control volume is fixed in shape by
assumption and the fluid is incompressible so we can not squeeze more in!). If the flow is one-dimensional then we can write

\[ \sum (V_i A_i)_{\text{out}} = \sum (V_i A_i)_{\text{in}} \quad \text{or} \quad \sum (Q)_{\text{out}} = \sum (Q)_{\text{in}} \]

**Example - Pipe Entrance Flow**

\[ u = U_{\text{max}} \left( 1 - \frac{r^2}{R^2} \right) \]

If the inlet flow is uniform and denoted \( U_0 \), what is \( U_{\text{max}} \)

\[ U_{\text{max}} = 2U_0 \]
3.4 Conservation of Linear Momentum

Newton’s second law for a fluid. It is a Lagrangian conservation law, as we have already discussed, hence we write

*Time-rate-of-change of the linear momentum of the system = Sum of the external forces*

or

\[
\frac{d}{dt} \int_{sys} \vec{v} \rho \, dV = \sum \vec{F}_{sys}
\]

which is true in an *inertial* (non-accelerating) reference frame.

To derive the conservation of linear momentum we substitute momentum \((m\vec{v})\) into the Reynolds Transport Theorem, which for \(B = m\vec{v}\) yields \(\beta = \vec{v}\), and gives us

\[
\frac{d}{dt} \int_{sys} \vec{v} \rho \, dV = \frac{d}{dt} \int_{C.V.} \vec{v} \rho \, dV + \int_{C.S.} \vec{v} \rho (\vec{v} \cdot \vec{n}) \, dA = \sum \vec{F}_{sys}
\]

But at a particular instant in time \(\sum \vec{F}_{sys} = \sum \vec{F}_{C.V.}\) and we write

\[
\frac{d}{dt} \int_{C.V.} \vec{v} \rho \, dV + \int_{C.S.} \vec{v} \rho (\vec{v} \cdot \vec{n}) \, dA = \sum \vec{F}_{C.V.}
\]

What is \(\sum \vec{F}_{C.V.}\)? It is the sum of surface forces and body forces.

- surface forces: Pressure acting on an area, shear stress acting on an area, ...
- body forces: Gravity, electromagnetic, ...

**Example of Determining \(\sum \vec{F}_{C.V.}\)**

Consider the nozzle:

\[
\sum \vec{F}_{C.V.} = 707 \text{ lbs}
\]
Review

- **Conservation of Mass** – In Reynolds Transport Theorem \( B = m, \beta = 1 \)

\[
\frac{d}{dt} \int_{C.V.} \rho \, dv + \int_{C.S.} \rho (\vec{v} \cdot \vec{n}) \, dA = 0
\]

If we have fixed control volume

\[
\int_{C.S.} \rho (\vec{v} \cdot \vec{n}) \, dA = 0
\]

And if the flow is also incompressible

\[
\int_{C.S.} (\vec{v} \cdot \vec{n}) \, dA = 0
\]

- **Conservation of Linear Momentum**

\[
\frac{d}{dt} \int_{C.V.} \vec{v} \rho \, dv + \int_{C.S.} \vec{v} \rho (\vec{v} \cdot \vec{n}) \, dA = \sum F_{C.V.}
\]

3.4.1 **Vector Nature of Equation and 1-D Form**

Keep in mind that this is a *vector equation*! The surface integrals lead to *momentum fluxes* \((\dot{m} \vec{v})\) across the surfaces. We hold to our established sign convention set by the positive outward facing normal vector to surface areas and find that the momentum fluxes out of the volume are positive while the momentum fluxes into the volume are negative.

- \( \vec{v} \cdot \vec{n} > 0 \) outward momentum flux

- \( \vec{v} \cdot \vec{n} < 0 \) inward momentum flux

regardless of the sign of \( \vec{v} \). The sign of \( \vec{v} \) establishes the sign of the momentum and hence you can have an inwardly directed negative momentum flux which yields, in 1-D, \( \dot{m} \vec{v} > 0 \).
If the flow is one-dimensional then \( \vec{v} \) and \( \rho \) are uniform at all inlets and outlets. Therefore we can write

\[
\int_{C.S.} \vec{v} \rho (\vec{v} \cdot \vec{n}) \, dA = \vec{v}_i (\rho_i v_i A_i) = \pm \dot{m} \vec{v}_i
\]

where the \( \pm \) arises from the dot product and determines whether the momentum flux is inward or outward. Hence our complete equation for a 1-D system is:

\[
\frac{d}{dt} \int_{C.V.} \vec{v} \rho \, d\gamma + \sum (\dot{m}_i \vec{v}_i)_{out} - \sum (\dot{m}_i \vec{v}_i)_{in} = \sum \vec{F}_{C.V.}
\]

Example – Conservation of Linear Momentum

What are \( F_x \) and \( F_z \) on the vane?

\[
F_x = \dot{m} V (\cos \theta - 1) = \rho AV^2 (\cos \theta - 1), \quad F_z = \dot{m} V \sin \theta = \rho AV^2 \sin \theta
\]

How does \( \vec{F} \) behave as a function of \( \theta \)?

\[
|\vec{F}| = 2 \dot{m} V \sin \frac{\theta}{2}
\]
3.5 Further discussion of Conservation of Linear Momentum and Control Volumes

Let’s revisit our nozzle example worked previously but this time let’s solve for $F_x$, the reaction force required to support the nozzle.

We have:

\[
\frac{d}{dt} = 0 \quad \Rightarrow \quad -\dot{m}_1 u_1 + \dot{m}_2 u_2 = -F_x + (P_0 + P_{atm})A_1 - P_{atm}A_1
\]

But $\dot{m}_1 = \dot{m}_2 = \dot{m}$, therefore

\[
F_x = \dot{m}(u_1 - u_2) + P_0 A_1
\]

What about forces on the nozzle only?

Let $R_x$ be the interaction between the fluid and the inside surface of the nozzle.

\[
F_x = (A_2 - A_1)P_{atm} + R_x
\]
What about forces on the fluid only?

\[ R_x = \dot{m}(u_1 - u_2) + (P_0 + P_{atm})A_1 - P_{atm}A_2 \]

Putting our above two equations together we should get the same result!

And we do! The determined forces depend strongly on the choice of control volume!