Review

- *Conservation of Linear Momentum*

\[
\frac{d}{dt} \int_{C.V.} \vec{v} \rho \, dV + \int_{C.S.} \vec{v} \rho (\vec{v} \cdot \vec{n}) \, dA = \sum \vec{F}_{C.V.}
\]

- a vector equation.

- **signs** - two locations, in and out across control surfaces \((\vec{v} \cdot \vec{n})\) term and sign of the velocity vector \((\vec{v})\).

- \(\sum \vec{F}\) on the C.V. accounts for pressure and shear stresses on the C.S., forces exposed by C.V., and any weights within C.V.

- Choose C.V. such that control surfaces are orthogonal to flow, therefore no shear stress on C.S. and the normal component is just the velocity.

- 1-D approximation is great - are there any issues? Yes - the momentum flux correction factor.

### 3.5.1 Variations From Uniform Flow - the Momentum Flux Correction Factor

As we have discussed we frequently assume that a flow is 1-D while we know in actuality it is not. Often this is an excellent assumption but sometimes the assumption is not as good and we may wish to correct for the effects of the dependence of the velocity on position. The terms that are affected are clearly the nonlinear terms so in the linear momentum equation the flux term is affected. If we wish to use the average velocity, \(\langle V \rangle\), as representative of a 1-D velocity equivalent to the 2-D velocity then we have

\[
\int_{CS} \vec{v} \rho (\vec{v} \cdot \vec{n}) \, dA = m_{out} \beta_{out} \langle V \rangle_{out} - m_{in} \beta_{in} \langle V \rangle_{in}
\]

where \(\beta \geq 1\) is known as the momentum flux correction factor and it accounts for the effect of the non-uniform velocity profile on the surface momentum flux. The definition
of the mean velocity is

\[ \langle V \rangle = \frac{\int \rho (\vec{v} \cdot \vec{n}) \, dA}{\rho A} \]

which for incompressible flows with the velocity vector normal to the control surface reduces to simply \( \langle V \rangle = Q/A \). Hence

\[ \dot{m} \beta \langle V \rangle = \int_{CS} \vec{v} \rho (\vec{v} \cdot \vec{n}) \, dA \]

and

\[ \beta = \frac{\int_{CS} \vec{v} \rho (\vec{v} \cdot \vec{n}) \, dA}{\dot{m} \langle V \rangle} \]

and if flow is normal to the entrance and exit planes of the C.V. (like it is in a pipe) then we can write:

\[ \beta = \frac{\int_{CS} v^2 \, dA}{A \langle V \rangle^2} \]

### 3.5.2 Ex. - Momentum Flux Correction Factor

Let’s revisit our pipe entrance flow example from the mass conservation section to see how significant the momentum flux correction factor can be:
3.6 Conservation of Angular Momentum

\[ \sum \vec{F} = m\vec{a} = \frac{d}{dt}(m\vec{v}) \]

Now, taking the moment of each side about some point \( A \) with position vector \( \vec{r} \)

\[ \sum \vec{r} \times \vec{F} = \frac{d}{dt}(\vec{r} \times m\vec{v}) = \frac{d}{dt}(\vec{r} \times \rho \vec{v}) \]

And

\[ \sum (\vec{r} \times \vec{F}_{ext}) = \frac{d}{dt} \int_{sys} (\vec{r} \times \vec{v})\rho \, dv \]

where \( H = \int_{sys} (\vec{r} \times \vec{v})\rho \, dv \) is the angular momentum of the system. Now we turn to the Reynolds Transport Theorem to find the conservation law for the angular momentum with \( B = H \) and \( \beta = \vec{r} \times \vec{v} \)

\[ \frac{dH_{sys}}{dt} = \frac{d}{dt} \left[ \int_{CV} (\vec{r} \times \vec{v})\rho \, dv \right] + \int_{CS} (\vec{r} \times \vec{v})\rho (\vec{v} \cdot \vec{n}) \, dA \]

For a non-deformable inertial control volume we have

\[ \sum (\vec{r} \times \vec{F}_{ext}) = \frac{\partial}{\partial t} \left[ \int_{CV} (\vec{r} \times \vec{v})\rho \, dv \right] + \int_{CS} (\vec{r} \times \vec{v})\rho (\vec{v} \cdot \vec{n}) \, dA \]

If we have 1-D inlets and outlets:

\[ \sum (\vec{r} \times \vec{F})_{ext} = \frac{\partial}{\partial t} \left[ \int_{CV} (\vec{r} \times \vec{v})\rho \, dv \right] + \sum (\vec{r} \times \vec{v})_{out}\dot{m}_{out} - \sum (\vec{r} \times \vec{v})_{in}\dot{m}_{in} \]

Note that the \((\vec{v} \cdot \vec{n})\) term leads to a positive or negative flux of angular momentum, as we found with the conservation of mass and linear momentum. However, angular momentum itself has a sign (as does linear momentum), which arises from the \( \vec{r} \times \vec{v} \) term so we must track two signs, which may cancel each other out. We note that our sign convention for the sign of angular momentum is the same as we used for torque, namely the right-hand-rule with a counter-clockwise sense yielding positive angular momentum and a clockwise sense yielding negative angular momentum. Hence the inward flux of clockwise angular momentum is positive and the outward flux of clockwise angular momentum is negative.
3.7 Angular Momentum Examples

3.7.1 Example 1 – Single-Arm Lawn Sprinkler

Consider the following single-arm lawn sprinkler

What is the optimal coordinate system and control volume?

A moving control volume ⇒ need absolute velocity, therefore

\[ V = \text{velocity relative to nozzle} + \text{velocity of nozzle} \]
\[ = \frac{Q}{A} - \Omega R \]

where \( A \) is the area of the pipe. Now, what is the solution?

\[ \Omega = \frac{Q}{AR} - \frac{T_0}{\rho QR^2} \]

Even if \( T_0 \), the retarding torque from friction, is zero, \( \Omega \) is finite.

What is \( \Omega \) when \( T_0 = 0? \) ⇒ \( \Omega = V_0/R \), where \( V_0 \) is the nozzle exit velocity

Why? ⇒ The absolute velocity out of the nozzle is zero!
3.7.2 Example 2 – Pipe section and Bracket Torque

Consider the following pipe section and supporting bracket:

What is the reaction torque at the bracket wall mounting point (A)? You may assume the fluid in the pipe is constant density and the surrounding environment is at atmospheric pressure.

How do we draw the control volume?

What is the equation?

What is \( \sum (\vec{r} \times \vec{F})_{CV} \)?

What are the pressure forces?

\[
T_A = h_2(P_2 A_2 + \dot{m} V_2) - h_1(P_1 A_1 + \dot{m} V_1) \quad \text{or since } \dot{m} = \rho Q = \rho A_1 V_1 = \rho A_2 V_2
\]

\[
T_A = h_2 A_2(P_2 + \rho V_2^2) - h_1 A_1(P_1 + \rho V_1^2)
\]