Review

• **Head Form of the Energy Equation**

\[
\left( \frac{P}{\gamma} + \frac{v^2}{2g} + z \right)_{out} = \left( \frac{P}{\gamma} + \frac{v^2}{2g} + z \right)_{in} - h_f
\]

where \( h_f \) is the friction head loss.

• A special case of the Energy Equation – The Bernoulli Equation

\[
\frac{P}{\gamma} + \frac{v^2}{2g} + z = h_0
\]

Can think of it as the inviscid (frictionless) conservation of energy equation (i.e., \( h_f = 0 = h_s \)).

• Stagnation pressure – the sum of the static pressure and the dynamic pressure

\[
P_s = P + \rho \frac{v^2}{2}
\]

3.14 Energy and the Hydraulic Grade Line

As we have seen we can write the head form of the Energy equation as

\[
\frac{P}{\gamma} + \frac{v^2}{2g} + z = H = \text{Energy Grade Line (EGL)}
\]

In the case of Bernoulli flows the energy grade line is simply a constant since by assumption energy is conserved (there is no mechanism to gain/lose energy). For other flows it will drop due to frictional losses or work done on the surroundings (e.g., a turbine) or increase due to work input (e.g., a pump). Note that this is the head that would be measured by a Pitot tube.

We can also write

\[
\frac{P}{\gamma} + z = \text{Hydraulic Grade Line (HGL)}
\]

and we see that the HGL is due to static pressure – the height a column of fluid would rise due to pressure at a given elevation or in other words the head measured by a static pressure tap or the **piezometric head**.
Example – Venturi Flow Meter

Consider

\[ Q = A_2 V_2 = A_2 \left( \frac{2g\Delta h}{1 - \left( \frac{A_2}{A_1} \right)^2} \right)^{\frac{1}{2}} \]

3.15 Variations From Uniform Flow

As we have discussed we frequently assume that a flow is 1-D while we know in reality it is not. Often this is an excellent assumption, but sometimes we may wish to correct for the effects of the dependence of the velocity on position. The term that is effected in the energy equation is the flux term. If we wish to use the average velocity, \( \langle V \rangle \), as representative of a 1-D velocity equivalent to the 2-D velocity then we have

\[
\int_{CS} \frac{v^2}{2} \rho (\vec{v} \cdot \vec{n}) dA = \dot{m} \left( \frac{\alpha_{out}}{2} \langle V \rangle_{out}^2 - \frac{\alpha_{in}}{2} \langle V \rangle_{in}^2 \right)
\]

where \( \alpha \) is known as the kinetic constant and it accounts for the effect of the non-uniform velocity profile on the surface flux of energy. The definition of the mean velocity is

\[
\langle V \rangle = \frac{\int \rho (\vec{v} \cdot \vec{n}) dA}{\rho A}
\]

which for incompressible flows with the velocity vector normal to the control surface reduces to simply \( \langle V \rangle = Q/A \). Hence

\[
\dot{m} \frac{\alpha_{out}}{2} \langle V \rangle^2 = \int_{CS} \frac{v^2}{2} \rho (\vec{v} \cdot \vec{n}) dA
\]
and

\[ \alpha = \frac{\int_{C_S} v^2 p (\vec{v} \cdot \vec{n}) \, dA}{\dot{m} \langle V \rangle^2} \]

Example

Consider the Laminar flow through a pipe sitting in a uniform velocity field in water:

What is \( \Delta P \)?

3.15.1 Frictional Effects

If we have abrupt losses, say at a contraction, a simple way of accounting for this is through a discharge coefficient. We can write a modified form of Torricelli’s formula for incompressible flow

\[ \langle V \rangle = \frac{Q}{A} = C_d \sqrt{2gh} \]

where \( C_d \) is the discharge coefficient and is 1 for frictionless (inviscid) flow and can range down to about 0.6 for flows strongly effected by friction. Note we can handle non-uniform (violation of 1-D assumption) flow effects with a \( C_d \) as well.
3.15.2 Vena Contracta Effect

For a flow to get around a sharp corner there would need to be an infinite pressure gradient, which of course does not happen. Hence if the boundary changes directions too rapidly at an exit, the flow separates from the exit and forms what is known as a *vena contracta*.

Clearly \( A_j/A \leq 1 \). For a round sharp-cornered exit the coefficient is \( C_c = A_j/A = 0.61 \) and typical values of the coefficient fall in the range \( 0.5 \leq C_c \leq 1.0 \).

3.16 Cavitation and the Bernoulli Equation

Consider the following flow geometry:
If we assume over such a short section friction is negligible then, given the constant \( z \), the Bernoulli equation reduces to

\[
P_0 = P + \frac{\rho v^2}{2}
\]

Therefore high pressure occurs when the velocity is low and as the velocity increases the pressure drops. Plotting the variation of the dynamic and static pressure we find that the pressure might fall below the vapor pressure of the fluid and hence cavitate. Thus in general we should be concerned about cavitation any time we are dealing with relatively high velocities.