Review

- **Vena Contracta** \( C_c = A_j / A = 0.61 \) for flow under a sluice gate.

- **Frictional effects and unknown non-uniform velocity profile** \( \Rightarrow \) discharge coefficient, \( C_d \) \( \Rightarrow \) \( V = C_d \sqrt{2gh} \).

- **Cavitation** – high velocity leads to low pressure, potential for cavitation.

- **Conservation of mass** \( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 = \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} \)

- **Incompressible Criteria** \( \text{Ma} \sim 0.3 \) \( \Rightarrow \) air \(|\vec{u}| < 100\ m/s\), water \(|\vec{u}| < 440\ m/s\)

- **Incompressible form of conservation of mass** \( \nabla \cdot \vec{u} = 0 \) \( \Rightarrow \) 2-D water waves – given \( u \) can determine \( w \).

\( \rightarrow \) continuity shows up often to close equations (solve for remaining unknown)!

### 4.3 Rotation Rate, Vorticity, and Irrotational Flow

The rotation rate, \( \Omega_y \), is given by

\[
\Omega_y = \frac{1}{2} \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right)
\]

We define vorticity as twice the rotation rate and hence

\[
\omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}
\]
Ex. - water waves again

Recall the equations for a 2-D water wave:

\[
\begin{align*}
    u(x, z, t) &= a \sigma \cos (kx - \sigma t)e^{kz} \\
    w(x, z, t) &= a \sigma \sin (kx - \sigma t)e^{kz}
\end{align*}
\]

Are water waves rotational?

\section{4.4 Acceleration Revisited - The Eulerian Perspective}

We define the acceleration of a particle or fluid parcel as

\[
\begin{align*}
    a_x &= \frac{du}{dt}, \quad a_y = \frac{dv}{dt}, \quad a_z = \frac{dw}{dt}
\end{align*}
\]

where

\[
\begin{align*}
    u &= \frac{dx}{dt}, \quad v = \frac{dy}{dt}, \quad w = \frac{dz}{dt}
\end{align*}
\]

clearly this is a Lagrangian description – we are describing the acceleration of a fluid parcel or particle which is moving with the flow. It is also apparent that the Eulerian and Lagrangian descriptions of velocity are identical, namely the time rate of change of position with respect to some point. Now what is the acceleration in an Eulerian reference frame?

We write the velocity as

\[
\begin{align*}
    u &= u(x, y, z, t), \quad v = v(x, y, z, t), \quad w = w(x, y, z, t) \quad \text{and} \quad \vec{u} = u\vec{i} + v\vec{j} + w\vec{k}
\end{align*}
\]
and by the chain rule we have

\[
 a_x = \frac{du(x, y, z, t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial t}
\]

\[
 = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}
\]

\[
 = \frac{\partial u}{\partial t} + \vec{u} \cdot \nabla u = \frac{Du}{Dt}
\]

where we recall that \( \frac{D}{Dt} \) is the \textit{substantial derivative}. Similarly

\[
 a_y = \frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}
\]

\[
 = \frac{\partial v}{\partial t} + \vec{u} \cdot \nabla v = \frac{Dv}{Dt}
\]

\[
 a_z = \frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}
\]

\[
 = \frac{\partial w}{\partial t} + \vec{u} \cdot \nabla w = \frac{ Dw}{Dt}
\]

Therefore,

\[
 \vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k} = \frac{Du}{Dt} \vec{i} + \frac{Dv}{Dt} \vec{j} + \frac{ Dw}{Dt} \vec{k} = \frac{D \vec{u}}{Dt}
\]

### 4.4.1 The Substantial Derivative – an Example

If a hiker starts at 10,000 feet on a 20,000 foot hike at 8:00am and is hiking to 14,000 feet, and a reasonable model of the atmosphere is the standard atmosphere, and a reasonable model of the diurnal effects of sun on temperature is \( \Theta = \Theta_{8am} + \alpha t \) where \( \alpha = 3^\circ \text{C}/\text{hr.} \). What velocity over ground must be sustained in order to experience a constant temperature?

0.8 mph.
4.5 Conservation of Linear Momentum

We start with the same infinitesimal control volume we used for conservation of mass

Therefore

\[ \sum F = \frac{\partial}{\partial t} \int_{CV} \vec{v} \rho \, dV + \sum_i (\dot{m}_i v_i)_\text{out} - \sum_i (\dot{m}_i v_i)_\text{in} \]

Now, the control volume is infinitesimal so we can take

\[ \frac{\partial}{\partial t} \int_{CV} \vec{v} \rho \, dV = \frac{\partial (\rho \vec{v})}{\partial t} \, dx \, dy \, dz \]

and the control surface momentum fluxes are

<table>
<thead>
<tr>
<th>Face</th>
<th>((\vec{m} \vec{v})_\text{in})</th>
<th>((\vec{m} \vec{v})_\text{out})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td>(\rho u \vec{v} , dy , dz)</td>
<td>((\rho u \vec{v})_\text{out})</td>
</tr>
<tr>
<td>(y)</td>
<td>(\rho v \vec{v} , dx , dz)</td>
<td>((\rho v \vec{v})_\text{out})</td>
</tr>
<tr>
<td>(z)</td>
<td>(\rho w \vec{v} , dx , dy)</td>
<td>((\rho w \vec{v})_\text{out})</td>
</tr>
</tbody>
</table>

Putting it all together (don’t forget the sign convention!)

\[ \sum F \, dx \, dy \, dz = \frac{\partial (\vec{v} \rho)}{\partial t} + \frac{\partial (\rho u \vec{v})}{\partial x} + \frac{\partial (\rho v \vec{v})}{\partial y} + \frac{\partial (\rho w \vec{v})}{\partial z} = \vec{v} \left[ \frac{\partial (\rho)}{\partial t} + \frac{\partial (\rho u \vec{v})}{\partial x} + \frac{\partial (\rho v \vec{v})}{\partial y} + \frac{\partial (\rho w \vec{v})}{\partial z} \right] + \rho \left[ \frac{\partial (\vec{v} \rho)}{\partial t} + \frac{\partial (u \vec{v})}{\partial x} + \frac{\partial (v \vec{v})}{\partial y} + \frac{\partial (w \vec{v})}{\partial z} \right] = \rho \frac{D\vec{v}}{Dt} \]

4.5.1 Forces Acting on Infinitesimal Control Volume

- Body force – e.g., gravity
- Surface forces – e.g., pressure (normal stress), shear stress

Gravity

\[ d\vec{F}_{\text{gravity}} = \rho \vec{g} \, dx \, dy \, dz \quad \text{we can write} \quad \vec{g} = -g \hat{k} \]
Surface Forces

Three components on each face – one surface normal and two orthogonal surface tangential.

The sign convention is + if outward in positive direction and − if outward in negative direction. Let’s consider just the $x$ components.

\[
\delta F_{Sx} = \left[ -\sigma_{xx} + \sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} \right] dy \, dz + \left[ -\tau_{yx} + \tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \right] dx \, dz
\]

\[
+ \left[ -\tau_{zx} + \tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \right] dx \, dy
\]

Therefore

\[
\frac{\delta F_{Sx}}{dx \, dy \, dz} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}
\]

Similarly

\[
\frac{\delta F_{Sy}}{dx \, dy \, dz} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}
\]

\[
\frac{\delta F_{Sz}}{dx \, dy \, dz} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}
\]

Now, putting it all together we have

\[\begin{align*}
\text{$x$ momentum} & \quad \rho \frac{Du}{Dt} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \\
\text{$y$ momentum} & \quad \rho \frac{Dv}{Dt} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \\
\text{$z$ momentum} & \quad \rho \frac{Dw}{Dt} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} - \rho g
\end{align*}\]
4.5.2 Inviscid Flow – The Euler Equations

If the flow is inviscid (frictionless) then there can be no shear stress ($\mu = \nu = 0$), thus $\tau = 0$ and we can define $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -P$. Making this substitution into our equations we arrive at the Euler Equations – the inviscid differential form of $\vec{F} = m\vec{a}$ for a fluid.

\[
\begin{align*}
x \text{ momentum} & \quad \frac{Du}{Dt} = -\frac{1}{\rho} \frac{\partial P}{\partial x} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \\
y \text{ momentum} & \quad \frac{Dv}{Dt} = -\frac{1}{\rho} \frac{\partial P}{\partial y} = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \\
z \text{ momentum} & \quad \frac{Dw}{Dt} = -\frac{1}{\rho} \frac{\partial P}{\partial z} - g = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}
\end{align*}
\]

or more compactly

\[
\frac{D\vec{v}}{Dt} = -\frac{1}{\rho} \nabla P + \vec{g}
\]

The Euler Equations are a non-linear partial differential equation (PDE). In their full form they are tough to solve! If you integrate the Euler Equations along a streamline you get the Bernoulli equation. If the flow is irrotational (the vorticity is zero everywhere) the Bernoulli constant on a streamline is a global constant (in the same irrotational region).

4.5.3 Viscous Flow – The Navier Stokes Equations

For a Newtonian fluid we know that stress is proportional to strain rate. It can be shown that:

\[
\begin{align*}
\sigma_{xx} & = -P + 2\mu \frac{\partial u}{\partial x} \\
\sigma_{yy} & = -P + 2\mu \frac{\partial v}{\partial y} \\
\sigma_{zz} & = -P + 2\mu \frac{\partial w}{\partial z}
\end{align*}
\]
and:

\[
\begin{align*}
\tau_{xy} &= \tau_{yx} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
\tau_{yz} &= \tau_{zy} = \mu \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \\
\tau_{xz} &= \tau_{zx} = \mu \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)
\end{align*}
\]

Substituting into our general momentum equations we have:

\[
\begin{align*}
x \text{ momentum} & \quad \rho \frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \\
y \text{ momentum} & \quad \rho \frac{Dv}{Dt} = -\frac{\partial P}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \\
z \text{ momentum} & \quad \rho \frac{Dw}{Dt} = -\frac{\partial P}{\partial z} + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) - \rho g
\end{align*}
\]

or

\[
\frac{D\vec{v}}{Dt} = -\frac{1}{\rho} \nabla P + \vec{g} + \nu \nabla^2 \vec{v}
\]

where we recall that \( \nabla^2 \) is the Laplacian and \( \nu = \mu/\rho \).

These are the Navier-Stokes equations. The exact equations for motion in a Newtonian fluid. We see that there are three equations and four unknowns \((u, v, w, P)\) hence we need one more equation to close the system. What is this equation? **Continuity!** Note that if the density is variable in the flow then we need what is known as an equation of state for the density, which often in environmental flows means also having transport equations for temperature (heat) and salinity since the equation of state is nothing more than the relationship between density and its variables of dependence which for environmental fluids will always include temperature and in coastal and oceanographic waters, salt.