Review

Momentum Equations

\[ \begin{align*}
  x \text{ momentum} & : \quad \rho \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right] = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \\
  y \text{ momentum} & : \quad \rho \left[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right] = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \\
  z \text{ momentum} & : \quad \rho \left[ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right] = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} - \rho g
\end{align*} \]

Euler’s Equation

Inviscid (e.g., frictionless), \( \Rightarrow \) \( \tau = 0, \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -P. \)

Therefore \( \frac{D\vec{V}}{Dt} = -\frac{1}{\rho} \nabla P + \vec{g} \) where we recall \( \vec{g} = -g\hat{z}. \)

Navier–Stokes Equations

\[ \frac{D\vec{V}}{Dt} = -\frac{1}{\rho} \nabla P + \vec{g} + \nu \nabla^2 \vec{V} \]

4.5.6 A Second Viscous Example – The Response of Cayuga Lake to Wind

Consider Cayuga lake as a simple rectangular long and narrow fish tank. Our picture is:
We make the following assumptions:

- $|\eta_{\text{max}}| \ll H$. Small $|\eta_{\text{max}}|$ leads to small $|u|$ and hence the non-linear terms have magnitude proportional to $|u^2|$ which is $\ll |u|$. Hence we will drop the non-linear terms.

- Steady state $\Rightarrow \partial / \partial t = 0$.

- Boundary and internal friction is negligible $\Rightarrow$ only the $\tau_{zx} = \tau_{xz}$ term is important as this carries the wind stress ($\tau_w$).

Thus our full equations reduce to:

\[ x \text{ momentum} \quad 0 = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial z} \quad (4.1) \]

\[ z \text{ momentum} \quad 0 = -\frac{\partial P}{\partial z} - \rho g + \frac{\partial \tau_{xz}}{\partial x} \quad (4.2) \]

But we also assume that the wind stress is constant $\Rightarrow \frac{\partial \tau_{xz}}{\partial x} = 0$.

Thus, we can write the $z$-momentum equation as

\[ \frac{\partial P}{\partial z} = -\rho g = -\gamma \]

The hydrostatic balance!

Now we recognize that we can integrate the $z$-momentum equation to find the pressure at any point $(x)$

\[ \int \frac{\partial P}{\partial z} = -\gamma z + C \]

where $C$ is an integration constant that may be a function of $x$ and $t$.

Now, applying the boundary conditions, which require that at $z = \eta \Rightarrow P = 0$, we have

\[ 0 = -\gamma \eta + C \quad \Rightarrow \quad C = \gamma \eta \]
and

\[ P(x, z) = \gamma [\eta(x) - z] \]

Substituting into the \(x\)-momentum equation we have

\[ \frac{\partial P}{\partial x} = \gamma \frac{\partial \eta}{\partial x} = \frac{\partial \tau_{xx}}{\partial z} \]

Note that this is independent of depth - e.g., it is constant over the depth at a horizontal location \(x\) (as required by hydrostatics).

And integrating in \(z\)

\[ \gamma \int_{-H}^{\eta} \frac{\partial \eta}{\partial x} \, dz = \int_{-H}^{\eta} \frac{\partial \tau_{xx}}{\partial z} \, dz \]

Therefore

\[ \gamma (\eta + H) \frac{\partial \eta}{\partial x} \bigg|_{-H}^{\eta} = \tau_{xx}(\eta) - \tau_{xx}(-H) = \tau_w - 0 \]

Recalling that \(\eta \ll H \Rightarrow \eta + H \approx H\) hence we have

\[ \gamma H \frac{\partial \eta}{\partial x} = \tau_w \]

Integrating in \(x\)

\[ \int_0^L \gamma H \frac{\partial \eta}{\partial x} \, dx = \int_0^H \tau_w \, dx \]

which gives

\[ \gamma H \eta = \tau_w x + C' \]

where \(C'\) is an integration constant. Rearranging we have:

\[ \eta = \frac{\tau_w x}{\gamma H} + \frac{C'}{\gamma H} = \frac{\tau_w x}{\gamma H} + C \]

where \(C\) is a new constant that has absorbed \(\gamma H\). How do we find \(C\)? Conservation of mass! Since \(\eta\) is simply a perturbation of the free surface, the area under the portion that goes below the mean (horizontal) position must exactly balance the portion that goes above the mean (horizontal) position of the surface, stated mathematically we have:

\[ \int_0^L \eta \, dx = 0 \]

Therefore

\[ 0 = \left[ \frac{\tau_w x^2}{H \gamma^2} + Cx \right]_0^L \]

And hence

\[ C = -\frac{\tau_w}{H\gamma} \frac{L}{2} \]

We have arrived at our final solution:

\[ \eta(x) = \frac{\tau_w}{\gamma H} \left( x - \frac{L}{2} \right) \]

We have found that if the wind blows steadily for long enough that a steady state is achieved and the surface of the lake simply tilts linearly in the direction of the wind! Clearly \( |\eta_{\text{max}}| \) occurs at \( x = 0 \) and \( x = L \). Let’s take a look at Cayuga Lake to get an idea of the magnitude of this tilting. For Cayuga Lake a very strong wind would be 20 m/s (44 mph, the remnants of Hurricane Isabel (2003) hit this speed and we seem to see it on average a few times each year). Researchers have studied the wind stress as a function of wind speed for lakes and found that:

\[ \tau_w = C_D \rho_{\text{air}} (U_{10})^2 \]

where \( U_{10} \) is the wind speed at 10 m above the lake and \( C_D \) is an empirically determined dimensionless constant that is about \( 1.5 \times 10^{-3} \). Taking \( U_{10} = 20 \text{ m/s} \) and the air temperature to be 20 °C we look up \( \rho_{\text{air}} = 1.2 \text{ kg/m}^3 \), hence \( \tau_w = 0.72 \text{ N/m}^2 \). Plugging this into our solution (taking Cayuga Lake’s average depth to be about 70 m and the length to be 60 km):

\[ \eta_{\text{max}} = \frac{0.72 \text{ N/m}^2}{70 \text{ m} \cdot 998 \text{ kg/m}^3 \cdot 9.81 \text{ m/s}^2 \cdot \frac{60,000 \text{ m}^2}{2}} = 3.2 \text{ cm} \]

3.2 cm! That does not sound like much but if you consider the size of the lake and moving all that water you realize just how much energy it is!