Review

Example – The Response of Cayuga Lake to Wind

- Friction important ⇒ Navier Stokes Equation
- Assume steady state and linear ⇒ $D\vec{V}/Dt \approx 0$.
- 2-D therefore only $x$ and $z$ momentum equations survive.
- Retain only the stress due to viscous effects from wind.

Hence:

$$0 = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xz}}{\partial z}$$

$$0 = -\frac{\partial P}{\partial z} - \rho g$$

We found $\eta(x) = \frac{\tau_W}{\gamma H} \left( x - \frac{L}{2} \right)$

5.1 Dimensional Analysis

Why? Control Volume and differential analysis based solutions don’t always exist (in fact they are the exception rather than the rule) ⇒ *Experimental Analysis!*

Goal: To be efficient with experiments ⇒ test the minimum number of relationships.

What is Dimensional Analysis?
A method for reducing the number of variables describing the physics ⇒ if $n$ dimensional variables are important we seek to reduce (condense) them to $k$ dimensionless variables.
5.2 Buckingham Pi Theorem

If $n$ dimensional variables are described by $j$ physical dimensions then $k = n - j$ independent dimensionless variables completely describe the physics.

While Buckingham is credited with the theorem, dimensional analysis had been used by Euler, Fourier, Rayleigh as well as Vaschy (French) and Riabouchinski (Russian).

Dimensional analysis is used in fluid mechanics, economics, plant biology, engineering in general, biomedical fields, the social sciences and other fields. It is a powerful fundamental technique!

5.3 Example: Drag Force on a Copepod

Consider a copepod, a small type of crustacean (about 1 mm in length) and one of the most abundant animals on the planet and foundation of the food web. Consider a copepod under motion in the Hudson Estuary:

Our engineering intuition tells us that

$$F = f(L, V, \rho, \mu)$$

where $F$ is the force on the copepod, $L$ is the typical length of the copepod, $V$ is the copepod’s velocity relative to the water, and $\rho$ and $\mu$ are the density and viscosity of the water, respectively.

Looking at the shape of a copepod, if we are interested in determining the drag force,
unless we grossly simplify its shape we can not easily directly measure the local state of stress at the boundaries or solve analytically for the stress using differential analysis. We could solve for the drag force using a control volume approach by looking at the momentum deficit in the wake or by directly measuring the drag force, but we would need to do this for a variety of different flows and copepod shapes (ages/types/scales). Hence we must test the effects of the four parameters experimentally.

Now, we want to determine the effect of $L, V, \rho,$ and $\mu$ on $F$. To do this we must hold three of the variables constant while varying one. Typically it might take 10 tests to find the dependence of $F$ on any one variable. Hence with 4 different variables we are faced with $10^4$ combinations to test! This is VERY time consuming and expensive!

Let’s apply Buckingham’s Pi theorem.

For our problem we have $n = 5$ dimensional variables ($F, L, V, \rho, \mu$). How many physical dimensions do they have?

- $F \sim [M^{1}L^{1}T^{-2}]$
- $L \sim [L]$
- $V \sim [LT^{-1}]$
- $\rho \sim [M^{1}L^{-3}]$
- $\mu \sim [M^{1}L^{-1}T^{-1}]$

Therefore we have three physical dimensions – $[M, L, T]$, hence $j = 3$ $\Rightarrow$ therefore $k = n - j = 2$. Hence we seek two dimensionless variables (known as Pi’s).

**First Method – Inspection**

Whenever we’re dealing with a moving fluid we are almost guaranteed to have the most well known of the dimensionless numbers in fluid mechanics show up – the *Reynolds number* (abbreviated Re):

$$\text{Re} = \frac{\rho VL}{\mu}$$

We’ll discuss the meaning of dimensionless numbers next lecture but essentially the Reynolds number tells us if a flow is turbulent (dominated by inertia) or laminar (dominated by viscosity).

Looking at our list of dimensional variables we see that we have already used 4 of them, the only unused dimensional variable is $F$. Hence we next seek to non-dimensionalize $F$. 
\[ F \sim [M L T^{-2}] \]

We immediately see that we can eliminate the time dependence by dividing \( F \) by \( V^2 \) and then remove the mass dependence by dividing by \( \rho \). After checking the remaining dimensions we find we only have a \( L^2 \) left so we divide through by \( L^2 \). Thus we have our second Pi:

\[ C_f = \frac{F}{\rho V^2 L^2} \]

Therefore we can now write:

\[ \frac{F}{\rho V^2 L^2} = g \left( \frac{\rho V L}{\mu} \right) \quad - \text{or} - \quad C_f = g(Re) \]

We have reduced our problem to a function of 1 variable - hence we can find the functional form with approximately 10 experiments! What a simplification, just by paying attention to the dimensions!

We note that function \( g \) is different than function \( f \) but still captures all the physics.

Hence, to test the drag force all we have to do is vary the Reynolds number which we can vary simply by changing the velocity of the flow (say in a flume).

**Benefits**

- Simplification \( \Rightarrow \) Less expensive and faster to test.

- Suggests simpler ways of writing equations (analytically) \( \Rightarrow \) Helps to see simpler theories to test.

- Can test potential designs at smaller scale by holding appropriate dimensionless numbers constant (e.g., we could test a larger model copepod at lower velocity and still be at the same Reynolds number and then expect the same drag).
Second Method – Formal Buckingham Pi Theory

1. List all relevant variables in problem - this is the hardest part!

2. List basic dimensions (M, L, T, Θ)

3. Determine number of Pi’s \( n - j = k \). Note that usually \( j \) is the number of variables but it can be less if certain variables appear always in the same grouping (see example 5.5 in the textbook for an example of this).

4. Pick \( j \) scaling variables (keep these simple - they will appear in all of your Pi's).
   In general you will likely want to build your Pi’s around certain variables (e.g., force and viscosity in the current example) and these should not be used as scaling variables.

5. Take your scaling variables and add one of the remaining variables to form a product of all variables.

6. Determine the exponents for each dimensional variable to ensure a dimensionless group (Pi).

7. Write down the dimensionless relationship amongst the variables.

Hence, working our example we choose the scaling variables to be \( \rho, V, L \). Thus:

\[
\Pi_1 = \rho^a V^b L^c \mu^1 = \left[ \frac{M}{L^3} \right]^a \left[ \frac{T}{L} \right]^b \left[ \frac{M}{LT} \right]^c = [MLT]^0
\]

Solving for \( a, b, c \):

\[
\begin{align*}
[M] & \quad a + 1 = 0 \quad \Rightarrow \quad a = -1 \\
[T] & \quad -b - 1 = 0 \quad \Rightarrow \quad b = -1 \\
[L] & \quad -3a + b + c - 1 = 0 \quad \Rightarrow \quad c = -1
\end{align*}
\]

therefore

\[
\Pi_1 = \rho^{-1} V^{-1} L^{-1} \mu^1 = \frac{\mu}{\rho V L} = Re^{-1}
\]
Of course this is a dimensionless quantity so we can just as easily work with the reciprocal (which we would have found had we taken the power of $\mu$ to be $-1$) which is the Reynolds number.

$$\Pi_2 = \rho^a V^b L^c F^1 = \left[ \frac{M}{L^3} \right]^a \left[ \frac{L}{T} \right]^b [L]^c \left[ \frac{ML}{T^2} \right]^1 = [M LT]^0$$

Solving for $a, b, c$:

$$[M] \quad a + 1 = 0 \quad \Rightarrow \quad a = -1$$
$$[T] \quad - b - 2 = 0 \quad \Rightarrow \quad b = -2$$
$$[L] \quad - 3a + b + c + 1 = 0 \quad \Rightarrow \quad c = -2$$

therefore

$$\Pi_2 = \rho^{-1} V^{-2} L^{-2} F^1 = \frac{F}{\rho V^2 L^2}$$

Just as before.

It is always interesting to consider what term is non-dimensionalizing our non-scaling variable ($F$ in this case) – $\rho V^2 L^2$. What is this? Well, $\rho V^2$ is the dynamic pressure (recall from the pressure form of the Bernoulli or Energy Equation) and of course $L^2$ is just an area so we are non-dimensionalizing by the dynamic pressure force – e.g., the force due to the flowing fluid.