Review

Similitude

Total similarity \( \Rightarrow \Pi_{1m} = \Pi_{1p}, \Pi_{2m} = \Pi_{2p}, \ldots, \Pi_{nm} = \Pi_{np} \) where the subscripts \( m \) and \( p \) indicate model and prototype, respectively.

Often can not achieve total similarity. We speak of

- Geometric Similarity – length scale ratios the same – e.g., aspect ratios, roughness ratios.
- Kinematic Similarity – velocity scale ratios the same – e.g., the Froude number
- Dynamic Similarity – force scale ratios the same – e.g., the Reynolds number
- We must be aware of model constraints bringing new physics (not present in the prototype scale) – for example, cavitation or compressibility effects may show up at model scale if the required velocity ratios are large.

6.1 Viscous Flow in Pipes

In general we speak of any flow completely bounded by walls as internal hence pipe flow is a specific form of internal flow. After investigating a few specific internal flows we will turn our attention to external flows - flows that are unbounded, at least in one dimension.

Initial Assumptions for Pipe Flow

- Round pipe. We will discuss other shapes and how to handle them but unless specifically told otherwise assume the pipe is round.
- Full pipe. Partially filled pipes are driven by gravity and we will discuss them later in the course.
**Laminar and Turbulent Flow - The Reynolds Number**

As discussed the Reynolds number expresses the ratio of inertial forces to viscous forces. For small Re viscous forces dominate and the flow is said to be laminar. In laminar flow the velocity at a point (in steady flow) is essentially constant (small fluctuations will damp out immediately - they are dissipated by viscous action).

At the other limit, large Re, inertial forces dominate and viscous effects are negligible. Thus if a fluctuation occurs in the velocity it will not damp out. It turns out that due to the nonlinearity in the acceleration terms in the Navier-Stokes equations the velocity field becomes more and more unstable (which is another way of saying it becomes prone to fluctuations or variations) as the Reynolds number increases. At high Re the flow is fully turbulent and the velocity and pressure fields have what appear to be random fluctuations (you’ve seen them on the ADV in Labs #3 & #4 - not the large oscillations in Lab #4 but the small oscillations). These fluctuations may be anywhere from 1% to more than 15% of the mean velocity.

At modest Reynolds number, the flow may oscillate between laminar and turbulent states. This is labeled *transition*.

Osborne Reynolds first demonstrated that the Reynolds number in fact characterizes the state of the flow – laminar, transition, and fully turbulent. The exact values of the Reynolds number that define these three regions vary depending on the geometry under study. For smooth walled circular pipe flow we have

- Re < 2100 – Laminar
- Re > 4000 – Turbulent
- 2100 < Re < 4000 – Transition

where for circular pipes we define Re as $Re_D$

$$Re_D = \frac{\rho V D}{\mu}$$
These values are approximate and depend on the smoothness of the pipe wall and the "quietness" of the initial flow (e.g., if the pump supplying the pipe is vibrating the pipe turbulence is apt to set in at lower Re).

What does turbulence look like?

Reynolds’ famous experiment uses dye to visualize streaklines at the center of a pipe (you’ll get your own chance to perform Reynolds’ experiment in Lab #5 - our final lab). The streaklines in laminar and turbulent flow typically look like:

If you placed a velocity probe in the center of the pipe, capable of measuring all three components of velocity accurately in space and time, you would find something that looks like:
6.2 Entrance Region

Consider the entrance region for a typical pipe flow:

Recall that conservation of mass (incompressible flow) requires:

\[ Q = \int_A u \, dA = \text{constant} \]

Therefore the potential core (inviscid or frictionless portion of the flow) accelerates as the boundary layers (BL’s) develop.

The entrance length, the length it takes the boundary layers to reach the center of the pipe and merge into a fully developed flow (e.g., the profile becomes independent of distance down the pipe), must be \( \ell_e = f(D, \mu, U, \rho) \), therefore \( \Pi_1 = \ell_e/D \) and \( \Pi_2 = Re_D \).

Hence we can write

\[ \frac{\ell_e}{D} = g(Re) \]

Experiments show that:

- Laminar Flow \( \frac{\ell_e}{D} = 0.06 \, Re \) \( \Rightarrow \) \( \ell_{e\text{max}} \sim 120D \)

- Turbulent Flow \( \frac{\ell_e}{D} = 4.4 \, Re^{1/6} \) \( \Rightarrow \) \( \ell_{e\text{max}} \sim 30D \)
6.3 Pressure Distribution at the Entrance

Consider the entrance region, what happens to the pressure?

\[ \frac{\partial P}{\partial x} = -\frac{\Delta P}{\ell} = \text{constant} \]

6.4 Pipe Flow – From Dimensional Analysis

With dimensional analysis under our belt let’s apply it to pipe flow and see what we can learn. Let’s consider a horizontal pipe, pictured below.

Since no component of the flow direction is in line with gravity (and there is no free surface as the pipe is full) we can ignore gravity. Hence the only variables that can effect the flow are the velocity \( V \), the diameter \( D \), the viscosity \( \mu \), the pipe length, \( L \) and the density \( \rho \). What is being affected? The pressure! We expect that the viscous action at the walls will dissipate energy and induce a head loss, hence we must include \( \Delta P \). Thus we write:

\[ \Delta P = f(V, D, \mu, L, \rho) \]

Clearly \( k=6, \ r=3, \) and we must have 3 \( \Pi \)’s. By visual inspection we found \( \Pi_2 = \text{Re}_D \), \( \Pi_3 = L/D \), and recalling from the Bernoulli equation that we can form the dynamic
pressure as $\frac{1}{2}\rho V^2$ we have $\Pi_1 = \Delta P/(\frac{1}{2}\rho V^2) = \text{Eu} = \text{Euler Number}$. Hence we now write:

$$\text{Eu} = g\left(\text{Re}_D, \frac{L}{D}\right)$$

This seems to be as far as we can go. However, if we consider steady state then the inertial terms are all zero and we should have a solution that is independent of density, if this is the case we can write

$$\Delta P = f'(V, D, \mu, L)$$

Now we have $k - r = 2$ with $\Pi_2 = L/D$, therefore, $\Pi_1 = \Delta PD/(\mu V)$, or

$$\frac{\Delta PD}{\mu V} = g'\left(\frac{L}{D}\right)$$

How does this help? Well, we expect that $\Delta P$ is directly related to the head loss (from the energy equation applied to the flow) and hence we expect that if the pipe length is doubled $\Delta P$ doubles. If this is true then we know our function $g'$ is just a constant (as this is the only way to preserve the linear relationship between $\Delta P$ and $L$). Hence we write

$$\frac{\Delta PD}{\mu V} = C\left(\frac{L}{D}\right)$$

Now, we expect that the constant ($C$) will depend on Reynolds number as our intuition is that turbulent and laminar flow are fundamentally different. Let’s rewrite our equation solving for $\Delta P$

$$\Delta P = C\mu V \frac{L}{D^2}$$

Now, recalling our original solution that involves Eu, Re, and the aspect ratio, we divide through by $\frac{1}{2}\rho V^2$ and get

$$\frac{\Delta P}{\frac{1}{2}\rho V^2} = C' \frac{\mu}{\rho V D} \frac{L}{D}$$

where $C'$ is a new constant that has absorbed a factor of 2 but may still be a function of Reynolds number (despite Re showing up formally in the equations the constant still may depend on Re as well). Hence we have:

$$\text{Eu} = \frac{C' L}{\text{Re}_D D}$$
All that remains is to determine the constant $C'$. It turns out that pipe flow yields an analytic solution to the Navier-Stokes Equations from which $C'$ can be found (we may work this as an example of Navier-Stokes analysis during Wednesday’s review section if you do not come armed with enough questions). Alternatively, we could study the constant $C'$ via experiment. We would find that our analysis stands up to both laminar and turbulent flows, however the constant is in fact a function of $Re$. If the flow is laminar $C' = a$ constant $= 64$. In fact the ratio $C'/Re$ is given a particular name, the friction factor or the Darcy friction factor. We write

$$\mathcal{F} = \frac{64}{Re}$$

for laminar flow. Note that $\mathcal{F}$ in the above expression (the Darcy friction factor) is often written as $f$ and should not be confused with the $f$ used earlier as the general function relating $\Delta P$ to $V, D, \ldots$ and $L/D$. They are two completely different $f$’s.

### 6.5 Energy Perspective

$$\frac{P_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_L$$

But the velocity profile is constant, therefore

$$\alpha_1 \frac{V_1^2}{2g} = \alpha_2 \frac{V_2^2}{2g}$$

Hence

$$h_L = \frac{P_1 - P_2}{\gamma} + (z_1 - z_2) = \frac{\Delta P}{\gamma} + \Delta z$$
6.6 Example

(1) Which way is the flow going? (2) What is $h_L$? (3) What is $V$ (or $Q$)? (4) What is $Re_D$ – and is a laminar assumption valid?

We begin with the energy equation for fully developed pipe flow:

$$h_L = \frac{P_1 - P_2}{\gamma} + z_1 - z_2 = \frac{\Delta P}{\gamma} + \Delta z$$

Now, assume flow is up the pipe (from high pressure to low pressure) – then $\Delta z$ is in the negative direction:

$$\Delta z = -L \sin 40^\circ = -6.43 \text{m}$$

Therefore

$$h_L = \frac{1.0 \times 10^5 \text{Pa}}{(900 \text{ kg/m}^3) (9.81 \text{ m/s}^2)} - 6.43 \text{m} = 4.90 \text{m}$$

Since $h_L$ is positive our assumed flow direction (up the pipe) is correct.

To find $V$ we recall

$$Eu = \frac{64L}{Re_D D}$$

Therefore

$$\frac{\Delta P}{\frac{1}{2} \rho V^2} = \frac{64 \nu L}{VD}$$

and

$$\Delta P = \frac{32 \nu \rho V L}{D^2}$$

But $\Delta P$ is the pressure drop due to frictional effects and hence neglecting gravity, therefore

$$V = \frac{h_L g D^2}{32 L \nu} = \frac{(4.90 \text{m}) (9.81 \text{ m/s}^2) (0.06 \text{m})^2}{32 (10 \text{m}) (2.0 \times 10^{-4} \text{ m}^2/\text{s})} = 2.7 \text{m/s}$$

Therefore

$$Re_D = \frac{(2.7 \text{m/s}) (0.06 \text{m})}{2.0 \times 10^{-4} \text{ m}^2/\text{s}} = 810$$

Therefore the laminar assumption is valid.
6.7 Turbulent Pipe Flow – Dimensional Analysis

We have the same analysis as laminar flow but now roughness, \( \epsilon \), is important. Therefore

\[
\Delta P = f(V, D, L, \epsilon, \mu, \rho)
\]

or

\[
Eu = g\left(Re_D, \frac{L}{D}, \frac{\epsilon}{D}\right)
\]

As before we expect a pipe of length \(2L\) will have a pressure drop of \(2\Delta P\) hence

\[
Eu = \frac{L}{D} g'\left(Re_D, \frac{\epsilon}{D}\right)
\]

From laminar analysis we found

\[
\frac{\Delta P D}{\frac{1}{2} \rho V^2 L} = f
\]

the friction factor

hence we write

\[
\Delta P = f \frac{L \rho V^2}{2}
\]

For laminar flow we found \(f = 64/\text{Re}\) – independent of \(\epsilon\).

For turbulent flow \(f\) must be determined from experiments. Luckily these experiments have been run by Nikuradse and Colebrook and to this day we rely on some form of interpolation of their data to estimate the friction factor for turbulent pipe flow.

We have

\[
h_L = \frac{\Delta P}{\gamma} + \Delta z
\]

Therefore for horizontal pipe flow we can write

\[
h_L = \frac{\Delta P}{\gamma} = f \frac{L V^2}{D 2g}
\]

This is known as the Darcy–Weisbach equation.

In general we have:

\[
\Delta P = \gamma \Delta z + f \frac{L \rho V^2}{D 2}
\]

which accounts for the hydrostatic contribution if there are changes in inlet and outlet elevation.
6.8 Pipe Flow – The Moody Chart

On the next page is the Moody chart (taken from White, 1999), which is Moody’s interpretation (and interpolation) of Colebrook’s data. We see the following

- Laminar flow \( f \) independent of \( \epsilon/D \)
- Fully turbulent flow \( f \) independent of \( \text{Re} \) – inertially dominated
- Turbulent flow at moderate \( \text{Re} \) – \( f \) a function of both \( \text{Re} \) and \( \epsilon/D \)

There are curve fits to the Moody chart, the most famous of which is the Colebrook formula:

\[
\frac{1}{\sqrt{f}} = -2.0 \log \left( \frac{\epsilon}{D} + \frac{2.51}{\text{Re}_D \sqrt{f}} \right)
\]

which is tough to work with as it depends on \( \sqrt{f} \) on both sides of the equation. Haaland has worked out an explicit relationship that is accurate to within 2%:

\[
\frac{1}{\sqrt{f}} = -1.8 \log \left[ \left( \frac{\epsilon}{D} \right)^{1.11} + \frac{6.9}{\text{Re}_D} \right]
\]
6.9 Explicit Solutions for 3-types of Pipe Flow - Swamee & Jain

Swamee and Jain, (J. of Hydraulics Division, Proc. ASCE, pp 657-664, May 1976), extended the concepts of Haaland to find explicit forms of the solutions for the three types of pipe flow, like Haaland, accurate to within 2% of the Moody diagram determined iterative solution. They are:

\[ h_L = 1.07 \frac{Q^2 L}{gD^5} \left\{ \log \left[ \frac{1}{3.7D} + 4.62 \left( \frac{\nu D}{Q} \right)^{0.9} \right] \right\}^{-2} \quad (3000 < \text{Re} < 3 \times 10^8; \ 10^{-6} < \frac{\epsilon}{D} < 0.01) \]

\[ Q = -0.965 \left( \frac{gD^5 h_L}{L} \right)^{0.5} \log \left[ \frac{1}{3.7D} + \left( \frac{3.17\nu^2 L}{gD^3 h_L} \right)^{0.5} \right] \quad (\text{Re} > 2000) \]

\[ D = 0.66 \left[ \epsilon^{1.25} \left( \frac{LQ^2}{gh_L} \right)^{4.75} + \nu Q^{9.4} \left( \frac{L}{gh_L} \right)^{5.2} \right]^{0.04} \quad (5000 < \text{Re} < 3 \times 10^8; \ 10^{-6} < \frac{\epsilon}{D} < 0.01) \]

6.10 Three Types of Pipe Flow Problems

<table>
<thead>
<tr>
<th>Type</th>
<th>Given</th>
<th>Find</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>(D, L, V, \rho, \mu, g)</td>
<td>(h_L) (\Rightarrow) (\Delta P)</td>
</tr>
<tr>
<td>II</td>
<td>(D, L, h_L, \rho, \mu, g)</td>
<td>(V) (or (Q))</td>
</tr>
<tr>
<td>III</td>
<td>(V) (or (Q)), (L, h_L, \rho, \mu, g)</td>
<td>(D)</td>
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The Moody chart is a head loss chart. It is assumed that you know all the variables listed in the *given* section of the *Type I* problem. If you do the solution is straightforward. If you do not (e.g., Types II and III) iteration will be required.
Example: Type I – Moody’s Original Problem For an asphalted cast iron pipe, find $h_L$ and $\Delta P$ given:

Example: Type II – Moody’s Problem but given $h_L = 4.5 \text{ ft}$, find $V$
Example: Type III – Moody’s Problem again! Given $Q = \frac{\pi D^2}{4} V = 1.18 \text{ ft}^3/\text{s}$, find $D$. 