Review

Minor Losses

All pipe fittings, direction, or diameter changes.

Loss coefficient

\[ K_L = \frac{\Delta P}{2 \rho V^2} \Rightarrow h_L = K_L \frac{V^2}{2g} \]

Non-Circular Pipes

- Laminar: \( f = 64/Re_D \pm 40\% \)
- Turbulent: \( f(Re_D, \epsilon/D_H) \Rightarrow \) Moody chart for \( f \pm 15\% \)

Bernoulli-Based Flow Metering

\[ Q = C_0 A_0 \sqrt{\frac{2\Delta P}{\rho (1 - \beta^4)}} \]

where \( \beta = d/D \) and \( C_0 = C_0(\beta, Re_D) \) is a coefficient determined by calibration.

7.1 External Flows

Bodies immersed in fluids:

- Planes, cars, ships, submarines, fish, birds, ...

Internal flows: the flow is dominated by the viscous boundary layers (pipe flow).

External flows: flows with viscous/shear confined to a near boundary region and inviscid flow away from the boundary.
Effect of Reynolds Number

7.2 Momentum Perspective

In boundary layer flows it is more or less standard to define the vertical (wall normal) coordinate as $y$ and the streamwise (wall parallel) coordinate as $x$ and we will adopt this convention here.
To apply conservation of linear momentum we need to know the fluxes across surfaces A, B, C and D.

A) 1-D inlet $\vec{V} \cdot \vec{n} = -U_\infty$
B) Streamline $\Rightarrow \vec{V} \cdot \vec{n} = 0$
C) 2-D outlet $\vec{V} \cdot \vec{n} = +u(y)$
D) Streamline $\Rightarrow \vec{V} \cdot \vec{n} = 0$

We also know:

The sum of the shear forces over the plate = the drag force in opposite direction as $U_\infty$

Pressure is uniform, therefore there is no net pressure force.

Steady state
$\Rightarrow$ Conservation of linear momentum gives:

$$\sum F_x = \rho \int u(\vec{V} \cdot \vec{n}) dA$$

$$-D = -\rho U_\infty^2 bh + \rho b \int_0^\delta u^2 dy$$

and hence

$$D = \rho U_\infty^2 bh - \rho b \int_0^\delta u^2 dy$$

Now we use conservation of mass to relate $h$ and $\delta$

$$\int_{CS} (\vec{V} \cdot \vec{n}) dA = 0$$

Therefore

$$U_\infty h = \int_0^\delta u dy$$

or

$$h = \frac{1}{U_\infty} \int_0^\delta u dy$$

Substituting back into our expression for $D$ we have:

$$D = \rho b U_\infty \int_0^\delta u dy - \rho b \int_0^\delta u^2 dy$$

$$D = \rho b \int_0^\delta u(U_\infty - u) dy \quad (7.1)$$
Equation 7.1 allows us to calculate the drag in terms of the deficit of momentum flux across the outlet of the control volume. Notice that if flow is inviscid, \( u = U_\infty \), and there is no drag.

Now, Kármán defined a momentum length scale, known as the momentum thickness \( \Theta \), as:

\[
\Theta(x) = \int_0^{\delta(x)} \frac{u}{U_\infty} \left(1 - \frac{u}{U_\infty}\right) dy
\]

where \( \Theta \) satisfies:

\[
\rho b \int_0^\delta u(U_\infty - u) dy = \rho b U_\infty^2 \Theta
\]

Then we can write

\[
D(x) = \rho b U_\infty^2 \Theta(x) \quad (7.2)
\]

Equation 7.2 is true for laminar or turbulent flow - we just need \( \tau(y) \).

To calculate the stress on the plate, we can continue by recalling that \( D(x) = b \int_0^x \tau_w(x) \, dx \), hence

\[
\frac{dD}{dx} = b \tau_w
\]

Therefore, with \( U_\infty = \) constant we have

\[
\frac{dD}{dx} = \rho b U_\infty^2 \frac{d\Theta(x)}{dx} = b \tau_w
\]

And finally

\[
\tau_w = \rho U_\infty^2 \frac{d\Theta(x)}{dx}
\]

This is known as the momentum integral equation for BL flow on a flat plate, valid for laminar and turbulent flow.
7.3 Example – von Kármán’s Laminar Boundary Layer Problem

Kármán modeled the laminar flat-plate boundary layer after the fully developed laminar pipe flow solution - namely a parabolic profile. He modeled the flow as:

\[ u(x, y) = U_\infty \left( \frac{2y}{\delta} - \frac{y^2}{\delta^2} \right) \]

What is \( \tau_w \)? What is \( \delta \)?

\[ \Theta = \int_0^\delta \left( \frac{2y}{\delta} - \frac{y^2}{\delta^2} \right) \left( 1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2} \right) dy = \frac{2}{15} \delta \]

Hence

\[ \tau_w = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} = \frac{2}{\delta} U_\infty = \rho U_\infty^2 \frac{2}{15} \frac{dx}{dx} \]

Therefore

\[ \delta \frac{d\delta}{dx} = 15 \frac{\mu}{U_\infty \rho} \frac{dx}{dx} \]

Integrating and assuming that \( \delta(0) = 0 \) we find

\[ \frac{\delta^2}{2} = \frac{15\nu x}{U_\infty} \]

Therefore we have found \( \delta \). In nondimensional form we have

\[ \frac{\delta}{x} = \sqrt{\frac{30\nu}{U_\infty x}} = \frac{5.5}{Re_x^{1/2}} \]

7.4 Skin Friction Coefficient

We can non-dimensionalize the wall stress by the dynamic pressure

\[ c_f = \frac{\tau_w}{\frac{1}{2} \rho U^2} \]

where we call \( c_f \) the skin friction coefficient (and it is clearly similar to the Euler number).

Hence, for von Kármán’s problem we have:

\[ c_f = \frac{2\mu U}{\frac{1}{2} \rho U_\infty^2 \delta} = \frac{4\mu}{\rho U_\infty \sqrt{\frac{30\nu x}{U_\infty}}} = \sqrt{\frac{8\nu}{15U_\infty x}} = \sqrt{\frac{8}{15Re_x}} \]
Therefore

\[ c_f \approx 0.73 \frac{Re_x^{\frac{1}{2}}}{Re_x} \]

As we will see soon, Kármán’s analysis is within 10% of the exact solution for laminar flows.

### 7.5 Boundary Layer Equations

Prandtl is credited with simplifying the Navier-Stokes equations to a tractable form suitable for boundary layers. He argued:

- \( v \ll u \)
- \( \frac{\partial}{\partial x} \ll \frac{\partial}{\partial y} \)

Hence the \( y \)-momentum equation reduces to:

\[ 0 = -\frac{\partial P}{\partial y} \quad \Rightarrow \quad P = P(x) \]

The outer flow is assumed constant (note if it is not constant the Euler equations can be invoked to solve for the pressure field) hence \( P = P(x) = \) a constant. Hence the \( x \)-momentum equations become

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\rho} \frac{\partial \tau}{\partial y} \]

where

\[ \tau = \mu \frac{\partial u}{\partial y} \]

and hence

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \]

This is the boundary layer form of the momentum equations. Note that the 2-D continuity equation closes the system of equations, i.e.,

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]
The Laminar Velocity Profile - Blasius' Solution

Prandtl could not solve the above equation but Prandtl’s student Blasius used a clever similarity variable technique and solve the boundary layer equations for laminar flow (see other fluids texts for details - see, good students are sharper than their professors!). A numerical solution to an ODE is required and a simple tabulated non-dimensional solution, known as the Blasius profile, is

\[
\begin{array}{cccc}
\eta & \frac{u}{u_\infty} & \eta & \frac{u}{u_\infty} \\
0 & 0 & 2.6000 & 0.7725 \\
0.2000 & 0.0664 & 2.8000 & 0.8115 \\
0.4000 & 0.1328 & 3.0000 & 0.8460 \\
0.6000 & 0.1989 & 3.2000 & 0.8761 \\
0.8000 & 0.2647 & 3.4000 & 0.9018 \\
1.0000 & 0.3298 & 3.6000 & 0.9233 \\
1.2000 & 0.3938 & 3.8000 & 0.9411 \\
1.4000 & 0.4563 & 4.0000 & 0.9555 \\
1.6000 & 0.5168 & 4.2000 & 0.9670 \\
1.8000 & 0.5748 & 4.4000 & 0.9759 \\
2.0000 & 0.6298 & 4.6000 & 0.9827 \\
2.2000 & 0.6813 & 4.8000 & 0.9878 \\
2.4000 & 0.7290 & 5.0000 & 0.9916 \\
\end{array}
\]

where \( \eta = y \left( \frac{U}{\nu x} \right)^{0.5} \). From the Blasius profile one finds:

\[
\frac{\delta}{x} = \frac{5.0}{\text{Re}_x^{1/2}} \quad \text{and} \quad c_f = \frac{0.664}{\text{Re}_x^{1/2}}
\]

Hence Kármán’s analysis, based on the assumed parabolic form, is within 10%!
7.6 Turbulent Flat Plate Boundary Layer

If we return to dimensional analysis the only parameters that can be important in the boundary layer are:

\[ \Pi = \phi(\rho, \nu, \tau_w, z) \]

Hence we have 5-3=2 dimensionless groups. We could select a Reynolds number based on the mean velocity and then nondimensionalize \( \tau_w \) to form the second group or we recall from drag coefficients that we can define a new velocity - the friction velocity as

\[ u^* = \sqrt{\frac{\tau_w}{\rho}} \]

and hence

\[ \Pi_1 = \frac{\Pi}{u^*} = u^+ \quad \text{and} \quad \Pi_2 = Re = \frac{u^*z}{\nu} = z^+ \]

Thus we can write

\[ u^+ = \psi(z^+) \]

Applying relevant local boundary conditions and taking the limits of viscous domination very near the wall and turbulence domination further from the wall, the functional form \( (\psi) \) of the solution to the above relationship can be found.

Very close to the wall where viscous effects are dominant the functional form is incredibly simple, \( u^+ = z^+ \).

A bit further from the wall where turbulence becomes dominant but the characteristic size of the turbulent eddies depends strongly on distance from the wall, the velocity profile form is:

\[ u^+ = \frac{1}{\kappa} \ln z^+ + C \]

where \( C \) is a constant generally taken to be 5.5 for high Reynolds number flows and \( \kappa \) is von Kármán’s constant, generally taken to be 0.41. This form of the solution is known as the log-law or law of the wall.
We speak of the turbulent boundary layer as having various regions:

\[
\begin{align*}
  z^+ &< 3.5 \quad \text{viscous sublayer} \\
  3.5 &< z^+ < 30 \quad \text{buffer layer or transition layer} \\
  30 &< z^+ \lesssim 200 \quad \text{log-law} \\
  z/\delta &> 0.2 \quad \text{Outer layer and defect law region}
\end{align*}
\]

The physical scale of \( z^+ \) is very small. As an example, in laboratory flumes a typical \( u^* \) is about 5\% of \( \overline{u} \) (in natural rivers a range of 5-10\% is typical). Hence for a flow of 20 cm/s \( u^* \approx 1 \text{ cm/s} \) \( \Rightarrow \) \( z^+ = 1 \) occurs at \( z = \nu/u^*=(0.01 \text{ cm}^2/\text{s})/(1 \text{ cm/s})=0.1 \text{ mm} \)!

Thus the viscous sub-layer is confined to the bottom 0.5 mm of the flow. Often it is reasonable to approximate the turbulent boundary layer over a flat plate as dominated by the log-law region. An example of this is shown in the plot on the next page. You will see that for \( y/\delta > 0.2 \) the log-law model underestimates the actual velocity profile (here represented by the direct numerical simulation of the Navier Stokes equation carried out by Spalart at \( Re_\theta = 1410 \) – this is the defect region and to do a better job we would need to model what is known as the velocity defect \( (U_\infty - \overline{u}) \).


An even simpler model of the turbulent boundary layer mean velocity profile over a flat plate is a power law formulation:

\[
\frac{\overline{u}}{U_\infty} = \left( \frac{y}{\delta} \right)^n
\]

where \( n \) is typically taken to be \( 1/7 \). In reality we expect \( n \) to depend on the Reynolds number and it does - it increases with increasing Re. Hence for low Re turbulence \( n = 1/6 \) is often a better fit as shown in the figure below.
Let’s conclude with a look at the flat plate boundary layer velocity profiles and the models. We have Blasius’s exact solution for Laminar flow. For turbulent flow to obtain an exact solution we must resort to a computational solution of the full Navier Stokes solutions. This calculation was performed by Philippe Spalart (1986). Direct simulation of a turbulent boundary layer up to $Re_\Theta=1410$. NASA Technical Memorandum 89407.

Note that Spalart’s calculation was for relatively low Re and hence the power law with best fit is about $n = 1/6$ and the constant $C$ in the log-law is 5.0. It is clear that the Blasius solution is very close to a simple parabola (e.g., Kármán’s solution). Also, for the inner region of the turbulent boundary layer the log-law works exceptionally well. Hence for lower elevations in the atmosphere (say less than a few hundred meters from the ground) the log-law works very well.

### 7.7 The Flat Plate Boundary Layer Velocity Profile
7.8 BL Example