Review

- *Flow through a contraction*
- *Critical and choked flows*
- *The hydraulic jump ⇒ conservation of linear momentum*

\[ \frac{y_2}{y_1} = -1 + \sqrt{1 + 8 \text{Fr}_1^2} \]

8.13 Rapidly Varied Flows – Weirs

8.13.1 Broad-Crested Weir

Consider the following flow:

This is known as the broad-crested weir which is characterized by:

- Sufficiently short that energy loss due to channel friction is negligible ⇒ \( h_L = 0 \) ⇒ Bernoulli’s equation.

- Sufficiently long horizontal section that hydrostatic flow is a reasonable approximation ⇒ pressure over horizontal section is hydrostatic.

Therefore our starting point is the Bernoulli equation

\[ E_1 = E_2 + h_w \Rightarrow \frac{V_1^2}{2g} + H + h_w = \frac{V_2^2}{2g} + h_w + y_2 \]
Aha! Since the flow upstream is subcritical and there is a region where $\frac{dy}{dx} \sim 1$ just upstream of the weir the flow over the horizontal section must be a control and hence $y_2 = y_c$ and $V_2^2 = V_c^2 = gy_c$. Therefore we have

$$\frac{V_1^2}{2g} + H = \frac{y_c}{2} + y_c = \frac{3}{2}y_c$$

Solving for $y_c$ we have

$$y_c = \frac{V_1^2}{3g} + \frac{2}{3}H \quad \approx \quad \frac{2}{3}H \quad \text{if} \quad \frac{V_1^2}{2g} \ll H$$

Therefore a reasonable approximation of the flow rate over a weir is

$$Q = by_cV_c = by_c\sqrt{gy_c} = b\sqrt{gy}\sqrt{3/2} = b\sqrt{3/2} = \left(\frac{2}{3}\right)^{3/2}b\sqrt{gH^3}$$

Now the reality is while the above is reasonable, there are energy losses and often broad-crested weirs are used as flow discharge measurement devices. Hence experimental calibration is often preferred and a weir discharge coefficient -- $C_d$ is experimentally determined, i.e.,

$$Q = C_d b\sqrt{gH^3}$$

The literature is full of different formulations such as equations 10.57 and 10.58 in your textbook which yields a maximal $C_d$ of about 0.54.

### 8.13.2 Sharp-Crested Weir

Now our picture looks like:

This is known as the sharp-crested weir. We can get a first-order solution by assuming
that the velocity field upstream of the weir is uniform and that the fluid flows horizontally over the weir and that the flow in the overflow section (known as the *nappe*) is a free jet and hence the pressure is atmospheric. Now our picture is:

Assuming that the energy losses are minimal and following our pictured streamline we have from the Bernoulli equation

\[
\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{u_2^2}{2g} + (h_w + y_2 - h)
\]

where \(u_2\) is the velocity at the streamline elevation over the weir and \(h\) is the distance from the free surface to the elevation of our streamline.

Now, since energy is conserved we know

\[
\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{V_2^2}{2g} + (h_w + y_2)
\]

and we can write:

\[
u_2 = \left[2g \left(h + \frac{V_1^2}{2g}\right)\right]^{\frac{1}{2}}
\]

Assuming a rectangular weir of width \(b\) we can find the flow rate by integrating:

\[
Q = \sqrt{2gb} \int_0^{y_2} \left(h + \frac{V_1^2}{2g}\right)^{\frac{1}{2}} dh
\]

Now, if \(h_w \gg y_2\) (which is often the case) the upstream energy is negligible and we can ignore the \(V_1^2\) term which simplifies the result of our integral too:

\[
Q = \frac{2}{3} \sqrt{2gy_2^3}
\]
very similar to our broad-crested weir result! Analogously we can write a weir coefficient to handle energy losses and geometry effects

\[ Q = C_{wrf} \frac{2}{3} \sqrt{2gy_2^4} \]

The book actually uses \( y_2 \approx \left( \frac{2}{3}H \right) \) and gets

\[ Q = C_d b \sqrt{gH^3} \]

Identical to the broad-crested weir except that the coefficient must differ. They give the coefficient in equation 10.56.

8.14 An Example Problem
8.15 Gradually Varied Flow  \( \frac{dy}{dx} \ll 1 \)

Unlike the previous example of a sill where we had \( \frac{dy}{dx} \ll 1 \) over a short enough reach that it was reasonable to assume \( h_L = 0 \) and hence use the Bernoulli form of the energy equation, for general gradually varied flow (GVF) the reaches are longer and we must include \( h_L \) that arises from frictional effects at the boundaries (e.g., include terms with Manning’s \( n \)). Therefore the general analysis of GVF combines the continuity equation with the energy equation. This leads to a first-order ordinary differential equation (ODE) for \( \frac{dy}{dx} \) which is readily solved numerically. An example of a standard software package to analyze GVF is the Army Corps of Engineers HEC-RAS (Hydraulic Engineering Center - River Analysis System). The analysis of these flows is beyond the scope of CEE 3310 but for those interested it is covered in CEE 3320(?).

8.16 Uniform Flow in a Partially Full Pipe

The geometry gives us:

\[
A = R^2 \left( \theta - \frac{\sin 2\theta}{2} \right); \quad P = 2R\theta
\]

Therefore \( R_h = \frac{R}{2} \left( 1 - \frac{\sin 2\theta}{2\theta} \right) \)

From Manning we have:

\[
V = \frac{\alpha}{n} \left[ \frac{R}{2} \left( 1 - \frac{\sin 2\theta}{2\theta} \right) \right]^{2/3} S_0^{1/2}
\]

Plotting \( V \) and \( Q \) as a function of \( \theta \):
\[ V_{\text{max}} = 0.718 \frac{\alpha}{n} R_h^{2/3} S_0^{1/2} \quad \text{at} \quad \theta = 128.7^\circ, \ y = 0.813D \]
\[ Q_{\text{max}} = 2.129 \frac{\alpha}{n} R_h^{8/3} S_0^{1/2} \quad \text{at} \quad \theta = 151.2^\circ, \ y = 0.938D \]

### 8.17 Efficient Uniform Flow Channels

We seek the lowest flow resistance \( \Rightarrow \) therefore wish to maximize \( R_h = A/P \). Therefore we wish to minimize \( P \) for a given \( A \).

The global most efficient shape is a half-full circular cross-section (the semi-circular channel).

How about for a trapezoidal channel? Consider:

\[ A = (y + y \cot \theta) b; \quad \mathcal{P} = b + 2y(1 + \cot^2 \theta)^{1/2} \]

Now we can solve for \( b \)

\[ b = \frac{A}{y(1 + \cot \theta)} \]
Therefore

\[ P = \frac{A}{y(1 + \cot \theta)} + 2y(1 + \cot^2 \theta)^{1/2} \]

Setting \( \frac{\partial P}{\partial y} = 0 \) for constant \( A \) and \( \theta \) we find:

\[ A = y^2 \left[ 2(1 + \cot^2 \theta)^{1/2} - \cot \theta \right] \]

\[ P = 4y(1 + \cot^2 \theta)^{1/2} - 2y \cot \theta \]

Therefore:

\[ R_h = \frac{1}{2}y \]

Aha! The most efficient section for an arbitrary trapezoid has \( R_h \) equal to one-half the depth.
General Topics Covered in CEE3310

- Viscosity and Shear Stress
- Hydrostatics – Pressure distribution in a fluid at rest, Forces on plane and curved surfaces, Manometry, Buoyancy
- Kinematics – Streamlines, Streaklines, Pathlines and the Substantial derivative
- Conservation of Mass, Continuity – Control Volume and Differential approach
- Conservation of Linear Momentum – Control Volume approach
- Conservation of Angular Momentum – Control Volume approach
- Conservation of Energy – Control Volume approach → Bernoulli Equation
- Dimensional Analysis
- Similitude and Modeling
- Pipe Flow – Laminar and Turbulent, Minor and Major losses
- Boundary Layer Flow – Laminar and Turbulent
- Drag
- Open Channel Flow – Manning’s equation, Specific Energy, Controls, The Hydraulic Jump, & Weirs