Chapter 3

Control Volume Analysis

3.1 Review

- Hydrostatic force on curved surfaces
- Pressure Prism
- Line of Action

3.2 Systems & Control Volumes

A system is a particular collection of matter separated from everything external by imaginary or real closed boundaries.
A systems’ mass is conserved. This is so fundamental in solid mechanics that it is not often written down:

\[ m = \text{const} \quad \text{or} \quad \frac{dm}{dt} = 0 \]

which is a fundamental law of mechanics, the conservation of mass. A system based analysis of fluid flows involves the Lagrangian reference frame, however, the system is moving and deforming—therefore it is hard to track its boundary! Consider turbulent flows, in these flows it is hard to even identify the boundaries!

A Control Volume (CV) is a defined volume in space, always identifiable, that may move and or deform but in an independent manner from the flow field. Mass, momentum and energy flows across the boundaries, known as control surfaces, that demarcate the control volume. This is an Eulerian type view (fixed size, location and shape, perhaps moving but at arbitrary velocity with respect to the flow itself).

We recognize that Newton’s second law, that unbalanced forces lead to accelerations

\[ \vec{F} = m \vec{a} = m \frac{d\vec{V}}{dt} = \frac{d}{dt} \left( m\vec{V} \right) \]

applies to systems (since it applies to a unique collection of mass, which are what was analyzed in solid mechanics) but in most cases we will not wish to follow a fluid system (e.g., a slug of water as it goes through, and ultimately out of, a hose) but instead will choose to define a control volume and describe the flow from the perspective of this fixed volume (e.g., we analyze the volume contained within the hose and we have flow in and out of this volume). Newton’s second law in fluid mechanics is known as the Conservation of Linear Momentum and it clearly has three component equations in the three Cartesian directions.

A third law of mechanics arises from unbalanced moments on systems

\[ \vec{M} = \frac{d\vec{H}}{dt} \quad \text{where} \quad \vec{H} = \sum \left( \vec{r} \times \vec{V} \right) \delta m \]

which basically says that unbalanced moments about the center of mass lead to rotation. In fluid mechanics this will lead to the the Conservation of Angular Momentum.
Finally, if heat $\delta Q$, is added to a system or work $\delta W$ is done by the system on the surroundings, the system energy must change according to the first law of thermodynamics

$$\delta Q - \delta W = dE \quad \text{or} \quad \dot{Q} - \dot{W} = \frac{dE}{dt}$$

which in fluid mechanics will lead to the \textit{Conservation of Energy}.

### 3.2.1 Volume and Mass Flow Rate

In control volume analysis we expect flows across surfaces.

What is the volume of flow across $S$ per unit time?

$$d\bar{v} = \delta t \bar{v} \cdot dA = v \delta t \cos \theta dA = (\bar{v} \cdot \bar{n}) dA \delta t$$

Now, we define $Q=\text{volume/time}=\text{volume flow rate}$. Then:

$$Q = \int_S \frac{d\bar{v}}{\delta t} = \int_S (\bar{v} \cdot \bar{n}) dA$$

Now, we define the normal vector $\bar{n}$ to be positive in the outward direction then

$$\bar{v} \cdot \bar{n} > 0 \quad \text{outflow}$$

$$\bar{v} \cdot \bar{n} < 0 \quad \text{inflow}$$

Further, if we multiply by the density, $\rho$, we have the \textit{mass flow rate}

$$\dot{m} = \int_S \rho (\bar{v} \cdot \bar{n}) dA$$

If $\rho$ is a constant then

$$\dot{m} = \rho Q$$

If we can assume the flow is one-dimensional then we can write

$$\dot{m} = \rho Q = \rho AV \quad \text{where } V = \text{the average velocity across } A$$
3.2.2 Volume & Mass Flow Rate – an Example

Consider the pipeline reducer:

If the pipe fluid is water with a mass flow rate of \( \dot{m} = 300 \text{ kg/s} \), what is \( Q_1, Q_2, V_1, V_2 \)?

\[ Q_1 = Q_2 = Q = 0.3 \text{ m}^3/\text{s}, \quad V_1 = 4.24 \text{ m/s}, \quad V_2 = 9.55 \text{ m/s} \]

3.2.3 Flow Rates Per Unit Area – the Flux

Consider the flow rate across any surface per unit surface area – this is known as the flux. Thus if we consider the volumetric flow rate per unit area this has the dimensions of:

\[ \bar{q} = \frac{[L^3]}{[T \cdot L^2]} = \frac{[L]}{[T]} = \text{a velocity!} \]
which in this case is the velocity normal to the surface. If we consider the mass flux (mass flow rate per unit area) this has the dimensions of:

\[ \vec{q}_m = \frac{[M]}{[T \cdot L^2]} \]

Note that flux is a vector quantity (in the direction normal to a surface). Often we speak of the total or net flux, which in the case of the volume and mass flow rates is just the volume and mass flow rates themselves.

### 3.3 The Reynolds Transport Theorem

We need a way to connect our systems approach to a control volume approach, let’s track a one-dimensional fixed system and control volume in the same flow and see if we can connect the two. Consider the following pipe expansion flow:

At \( t = t \), the system = control volume
At \( t = t_0 + dt \), the system = (control volume - I) + II

Let \( B \) be any fluid property (e.g., mass, velocity, energy ..., known as an extensive quantity) then we can define

\[ \beta = \frac{B}{m} \]

as the amount of \( B \) per unit mass, this is known as and intensive quantity

The total amount of \( B \) in the control volume can be found as

\[ B_{CV} = \int_{CV} \beta \rho \, dv \]

where \( \rho \, dv \) is seen to be an infinitesimal mass, \( \delta m \), within the CV
Now, we are interested in the time-rate-of-change of $B$ within the control volume so we have

$$\frac{dB_{CV}}{dt} = \frac{B_{CV}(t+dt) - B_{CV}(t)}{dt}$$  \hspace{1cm} (3.1)

$$= \frac{B_2(t+dt) - (\beta \rho d\mathcal{V})_{out} + (\beta \rho d\mathcal{V})_{in} - B_2(t)}{dt}$$  \hspace{1cm} (3.2)

$$= \frac{B_2(t+dt) - B_2(t)}{dt} - (\beta \rho AV)_{out} + (\beta \rho AV)_{in}$$  \hspace{1cm} (3.3)

The first term on the right-hand-side is the rate-of-change of $B$ within system 2 at the instant it occupies the control volume, which is the quantity we want to relate to the rate-of-change of $B$ within the control volume itself, hence we re-write the above:

$$\frac{dB_{CV}}{dt} = \frac{dB_{sys}}{dt} - (\beta \rho AV)_{out} + (\beta \rho AV)_{in}$$

or, invoking our expression for $B_{CV}$ above, we have

$$\frac{dB_{sys}}{dt} = \frac{d}{dt} \left( \int_{CV} \beta \rho d\mathcal{V} \right) + (\beta \rho AV)_{out} - (\beta \rho AV)_{in}$$

which we can write in a simplified form

$$\frac{dB_{sys}}{dt} = \frac{dB_{CV}}{dt} + \dot{B}_{out} - \dot{B}_{in}$$

For our simple 1-D constant control volume system we can write

$$\frac{dB_{sys}}{dt} = \frac{dB_{CV}}{dt} + \rho_{out} \beta_{out} Q_{out} - \rho_{in} \beta_{in} Q_{in}$$

This is the Reynolds Transport Theorem for a fixed control volume one input, one output system.

### 3.3.1 Generalized Reynolds Transport System

We can generalize our above result to arbitrary shaped control volumes as

$$\frac{dB_{sys}}{dt} = \frac{d}{dt} \left( \int_{CV} \beta \rho d\mathcal{V} \right) + \int_{CS} \rho \beta (\vec{v} \cdot \vec{n}) dA$$
And if the flow is steady we can write the time derivative of the control volume term as

\[
\frac{dB_{sys}}{dt} = \left( \int_{CV} \frac{\partial}{\partial t} (\beta \rho) dV \right) + \int_{CS} \rho \beta (\vec{v} \cdot \vec{n}) dA
\]

And if the control volume is moving at a velocity \( \vec{V}_{CV} \) an observer standing on the control volume would experience a relative velocity \( \vec{V}_r = \vec{v} - \vec{V}_{CV} \) and we can write our expression for the moving control volume as

\[
\frac{dB_{sys}}{dt} = \frac{d}{dt} \left( \int_{CV} \beta \rho dV \right) + \int_{CS} \rho \beta (\vec{V}_r \cdot \vec{n}) dA
\]

### 3.3.2 Simplifications

Consider:

Steady flow therefore \( \frac{\partial}{\partial t} = 0 \). Flow across a surface is a flux. Therefore the in/out terms in the above picture are flux terms.

The inlets and outlets are all 1-D, hence

\[
\int_S \rho \beta (\vec{v} \cdot \vec{n}) dA = \sum_{i=1}^{N} \rho_i \beta_i Q_i \frac{\vec{v} \cdot \vec{n}}{|\vec{v} \cdot \vec{n}|}
\]

\[
= \rho_1 \beta_1 V_1 A_1 + \rho_2 \beta_2 V_2 A_2 - \rho_3 \beta_3 V_3 A_3
\]