3.3 Review

- Line of Action – the direction of the resultant force vector, $\vec{F}_r$, which passes through the center of pressure, which we know is located below the centroid unless the surface is horizontal.

3.4 The Reynolds Transport Theorem

We need a way to connect our systems approach to a control volume approach, let’s track a one-dimensional fixed system and control volume in the same flow and see if we can connect the two. Consider the following pipe expansion flow:

At $t = t$, the system = control volume
At $t = t_0 + dt$, the system=$(control \text{ volume } - I)+II$

Let $B$ be any fluid property (e.g., mass, velocity, energy ..., known as an extensive quantity) then we can define

$$\beta = \frac{B}{m}$$

as the amount of $B$ per unit mass, this is known as and intensive quantity

The total amount of $B$ in the control volume can be found as

$$B_{CV} = \int_{CV} \beta \rho \, d\mathcal{V}$$

where $\rho \, d\mathcal{V}$ is seen to be an infinitesimal mass, $\delta m$, within the CV

Now, we are interested in the time-rate-of-change of $B$ within the control volume so we
have
\[
\frac{dB_{CV}}{dt} = \frac{B_{CV}(t + dt) - B_{CV}(t)}{dt} = \frac{B_2(t + dt) - (\beta \rho d\vec{v})_{out} + (\beta \rho d\vec{v})_{in} - B_2(t)}{dt} = \frac{B_2(t + dt) - B_2(t)}{dt} - (\beta \rho AV)_{out} + (\beta \rho AV)_{in}
\]

(3.1)

(3.2)

(3.3)

The first term on the right-hand-side is the rate-of-change of \( B \) within system 2 at the instant it occupies the control volume, which is the quantity we want to relate to the rate-of-change of \( B \) within the control volume itself, hence we re-write the above:

\[
\frac{dB_{CV}}{dt} = \frac{dB_{sys}}{dt} - (\beta \rho AV)_{out} + (\beta \rho AV)_{in}
\]

or, invoking our expression for \( B_{CV} \) above, we have

\[
\frac{dB_{sys}}{dt} = \frac{d}{dt} \left( \int_{CV} \beta \rho d\vec{v} \right) + (\beta \rho AV)_{out} - (\beta \rho AV)_{in}
\]

which we can write in a simplified form

\[
\frac{dB_{sys}}{dt} = \frac{dB_{CV}}{dt} + \dot{B}_{out} - \dot{B}_{in}
\]

For our simple 1-D constant control volume system we can write

\[
\frac{dB_{sys}}{dt} = \frac{dB_{CV}}{dt} + \rho_{out} \beta_{out} Q_{out} - \rho_{in} \beta_{in} Q_{in}
\]

This is the Reynolds Transport Theorem for a fixed control volume one input, one output system.

### 3.4.1 Generalized Reynolds Transport System

We can generalize our above result to arbitrary shaped control volumes as

\[
\frac{dB_{sys}}{dt} = \frac{d}{dt} \left( \int_{CV} \beta \rho d\vec{v} \right) + \int_{CS} \rho \beta (\vec{v} \cdot \vec{n}) dA
\]

And if the flow is steady we can write the time derivative of the control volume term as

\[
\frac{dB_{sys}}{dt} = \left( \int_{CV} \frac{\partial}{\partial t} (\beta \rho) d\vec{v} \right) + \int_{CS} \rho \beta (\vec{v} \cdot \vec{n}) dA
\]
And if the control volume is moving at a velocity $\vec{V}_{CV}$ an observer standing on the control volume would experience a relative velocity $\vec{V}_r = \vec{v} - \vec{V}_{CV}$ and we can write our expression for the moving control volume as

$$\frac{dB_{sys}}{dt} = \frac{d}{dt} \left( \int_{CV} \beta \rho \, dV \right) + \int_{CS} \rho \beta (\vec{V}_r \cdot \vec{n}) \, dA$$

### 3.4.2 Simplifications

Consider:

Steady flow therefore $\frac{\partial}{\partial t} = 0$. Flow across a surface is a flux. Therefore the in/out terms in the above picture are flux terms.

The inlets and outlets are all 1-D, hence

$$\int_S \rho \beta (\vec{v} \cdot \vec{n}) \, dA = \sum_{i=1}^{N} \rho_i \beta_i Q_i \frac{\vec{v} \cdot \vec{n}}{|\vec{v} \cdot \vec{n}|}$$

$$= \rho_1 \beta_1 V_1 A_1 + \rho_2 \beta_2 V_2 A_2 - \rho_3 \beta_3 V_3 A_3$$