3.5 Review

- **Buoyancy** – weight of fluid displaced by a body.

- **Stability** - overturning moment or restoring moment from force couple between $\vec{F}_B$ and $\vec{W}$.

- **System** – A particular collection of mass.

- **Control volume** – A particular volume in space.

- **Volume flow rate & Mass Flow Rate**

- **Flux** – the amount of stuff crossing a unit area in a unit of time

3.6 Conservation of Mass

We can derive the equation for conservation of mass, also known as the continuity equation, by simply letting $B = m$ in the Reynolds Transport Theorem. Therefore, $\beta = 1$ and we have

$$\frac{dm_{\text{sys}}}{dt} = \frac{d}{dt} \int_{C.V.} \rho \, d\forall + \int_{C.S.} \rho (\vec{v} \cdot \vec{n}) \, dA = 0$$

Now, if we have a fixed control volume then

$$\int_{C.V.} \frac{\partial \rho}{\partial t} \, d\forall + \int_{C.S.} \rho (\vec{v} \cdot \vec{n}) \, dA = 0$$

And if the flow is at steady state:

$$\int_{C.S.} \rho (\vec{v} \cdot \vec{n}) \, dA = 0$$

If we have one-dimensional inlets and outlets

$$\sum (\rho_i A_i V_i)_{\text{in}} = \sum (\rho_i A_i V_i)_{\text{out}}$$

or

$$\sum (\dot{m}_i)_{\text{in}} = \sum (\dot{m}_i)_{\text{out}}$$
3.6.1 A Note on Average Velocity

The velocity $V$ in $\dot{m} = \rho Q = \rho AV$ represents the spatially averaged velocity across $A$ and hence really $\langle V \rangle$ where

$$\langle V \rangle = \frac{\int_A \rho (\vec{v} \cdot \vec{n}) \, dA}{\rho A}$$

Only if $\rho$ and $V$ are not functions of $A$ does $\langle V \rangle = V$ (i.e., only if the flow is truly one-dimensional).

Example

$$\frac{dh}{dt} = \frac{\rho_1 A_1 V_1 + \rho_2 A_2 V_2}{\rho_w A}$$

3.6.2 Incompressible Flow, Fixed C.V.

For a fixed C.V. we have

$$0 = \int_{C.V.} \frac{\partial \rho}{\partial t} \, dV + \int_{C.S.} \rho (\vec{v} \cdot \vec{n}) \, dA$$

If the flow is incompressible then $\frac{\partial \rho}{\partial t} = 0$ and we have

$$0 = \int_{C.S.} \rho (\vec{v} \cdot \vec{n}) \, dA = \int_{C.S.} (\vec{v} \cdot \vec{n}) \, dA$$

Which states that whatever flows into the control volume must flow out since there is no change in storage in the control volume (the control volume is fixed in shape by
assumption and the fluid is incompressible so we can not squeeze more in!). If the flow is one-dimensional then we can write

\[ \sum (V_i A_i)_{out} = \sum (V_i A_i)_{in} \quad \text{or} \quad \sum (Q)_{out} = \sum (Q)_{in} \]

Example - Pipe Entrance Flow

\[ u = U_{max} \left( 1 - \frac{r^2}{R^2} \right) \]

If the inlet flow is uniform and denoted \( U_0 \), what is \( U_{max} \)

\[ U_{max} = 2U_0 \]

### 3.7 Conservation of Linear Momentum

Newton’s second law for a fluid. It is a Lagrangian conservation law, as we have already discussed, hence we write

*Time-rate-of-change of the linear momentum of the system = Sum of the external forces*
or

\[
\frac{d}{dt} \int_{\text{sys}} \bar{v} \rho \, d\mathcal{V} = \sum \bar{F}_{\text{sys}}
\]

which is true in an \textit{inertial} (non-accelerating) reference frame.

To derive the conservation of linear momentum we substitute momentum \((m\bar{v})\) into the Reynolds Transport Theorem, which for \(B = m\bar{v}\) yields \(\beta = \bar{v}\), and gives us

\[
\frac{d}{dt} \int_{\text{sys}} \bar{v} \rho \, d\mathcal{V} = \frac{d}{dt} \int_{\text{C.V.}} \bar{v} \rho \, d\mathcal{V} + \int_{\text{C.S.}} \bar{v} \rho (\bar{v} \cdot \bar{n}) \, dA = \sum \bar{F}_{\text{sys}}
\]

But at a particular instant in time \(\sum \bar{F}_{\text{sys}} = \sum \bar{F}_{\text{C.V.}}\) and we write

\[
\frac{d}{dt} \int_{\text{C.V.}} \bar{v} \rho \, d\mathcal{V} + \int_{\text{C.S.}} \bar{v} \rho (\bar{v} \cdot \bar{n}) \, dA = \sum \bar{F}_{\text{C.V.}}
\]

What is \(\sum \bar{F}_{\text{C.V.}}\)? It is the sum of surface forces and body forces.

- \text{surface forces}: Pressure acting on an area, shear stress acting on an area, ...
- \text{body forces}: Gravity, electromagnetic, ...

Example of Determining \(\sum \bar{F}_{\text{C.V.}}\).

Consider the nozzle:

\[
\sum \bar{F}_{\text{C.V.}} = 707 \text{ lbs}
\]