

7.1 Review

Minor Losses

All pipe fittings, direction, or diameter changes.

\[
K_L = \frac{\Delta P}{\frac{1}{2} \rho V^2} \quad \Rightarrow \quad h_L = K_L \frac{V^2}{2g}
\]

7.2 External Flows

Bodies emersed in fluids:

- Planes
- Cars
- Ships
- Submarines
- Fish
- Birds

In general the broad fields of hydrodynamics and aerodynamics cover external flows – flows with viscous/shear confined to a near boundary region and inviscid flow away from the boundary.

This contrasts with internal flows where the flow is dominated by the viscous boundary layers.
7.3 Momentum Perspective

In boundary layer flows it is more or less standard to define the vertical (wall normal) coordinate as $y$ and the streamwise (wall parallel) coordinate as $x$ and we will adopt this convention here.
A 1-D inlet $\vec{V} \cdot \vec{n} = -U_\infty$

B Streamline $\Rightarrow \vec{V} \cdot \vec{n} = 0$

C 2-D outlet $\vec{V} \cdot \vec{n} = +u(y)$

D Streamline $\Rightarrow \vec{V} \cdot \vec{n} = 0$, the sum of the shear forces over the plate = the drag force in opposite direction as $U_\infty$

Pressure is uniform, therefore there is no net pressure force. Steady state $\Rightarrow$ Conservation of linear momentum gives:

$$\sum F_x = \rho \int u(\vec{V} \cdot \vec{n}) dA$$

$$-D = -\rho U_\infty^2 b h + \rho b \int_0^\delta u^2 dy$$

and hence

$$D = \rho U_\infty^2 b h - \rho b \int_0^\delta u^2 dy$$

Now we use conservation of mass to relate $h$ and $\delta$

$$\int_{CS} (\vec{V} \cdot \vec{n}) dA = 0$$

Therefore

$$U_\infty h = \int_0^\delta u dy$$

or

$$h = \frac{1}{U_\infty} \int_0^\delta u dy$$

Substituting back into our expression for $D$ we have:

$$D = \rho b U_\infty \int_0^\delta u dy - \rho b \int_0^\delta u^2 dy$$

$$= \rho b \int_0^\delta u(U_\infty - u) dy$$

Now, define:

$$\Theta(x) = \int_0^{\delta(x)} \frac{u}{U_\infty} \left(1 - \frac{u}{U_\infty}\right) dy$$

Then we can write

$$D(x) = \rho b U_\infty^2 \Theta(x)$$
This is true for laminar or turbulent flow - we just need \( \tau(y) \).

We can continue by recalling that \( D(x) = b \int_0^x \tau_w(x) \, dx \) hence
\[
\frac{dD}{dx} = b \tau_w
\]
Therefore, with \( U_\infty = \) a constant we have
\[
\frac{dD}{dx} = \rho b U_\infty^2 \frac{d\Theta(x)}{dx} = b \tau_w
\]
And finally
\[
\tau_w = \rho U_\infty^2 \frac{d\Theta(x)}{dx}
\]
This is known as the momentum integral equation.

### 7.4 Example – von Kármán’s Laminar Boundary Layer Problem

Von Kármán modeled the laminar flat-plate boundary layer after the fully developed laminar pipe flow solution - namely a parabolic profile. He modeled the flow as:
\[
u(x, y) = U_\infty \left( \frac{2y}{\delta} - \frac{y^2}{\delta^2} \right)
\]
What is \( \tau_w \)? What is \( \delta \)?
\[
\Theta = \int_0^\delta \left( \frac{2y}{\delta} - \frac{y^2}{\delta^2} \right) \left( 1 - \frac{2y}{\delta} - \frac{y^2}{\delta^2} \right) \, dy = \frac{2}{15} \delta
\]
Hence
\[
\tau_w = \mu \frac{\partial u}{\partial y} \bigg|_{y=0} = \mu \frac{2}{\delta} U_\infty = \rho U_\infty^2 \frac{2}{15} \frac{d\delta}{dx}
\]
Therefore
\[
\delta \, d\delta = 15 \frac{\mu}{U_\infty \rho} \, dx
\]
Integrating and assuming that \( \delta(0) = 0 \) we find
\[
\frac{\delta^2}{2} = \frac{15 \nu x}{U_\infty}
\]
Therefore we have found $\delta$. In nondimensional form we have

$$\frac{\delta}{x} = \sqrt{\frac{30\nu}{U_\infty x}} = \frac{5.5}{Re_x^{1/2}}$$

### 7.5 Skin Friction Coefficient

We can non-dimensionalize the wall stress by the dynamic pressure

$$c_f = \frac{\tau_w}{\frac{1}{2} \rho U^2}$$

where we call $c_f$ the skin friction coefficient (and it is clearly similar to the Euler number). Hence, for von Kármán’s problem we have:

$$c_f = \frac{2\mu U}{\frac{1}{2} \rho U_\infty^2 \delta} = \frac{4\mu}{\rho U_\infty \sqrt{30\nu x} U_\infty} = \sqrt{\frac{8\nu}{15U_\infty x}} = \sqrt{\frac{8}{15Re_x}}$$

Therefore

$$c_f \approx \frac{0.73}{Re_x^{1/4}}$$

As we will see in a minute, Kármán’s analysis is within 10% of the exact solution for laminar flows.