8.1 Review

Drag & Lift \Rightarrow Laminar vs Turbulent Boundary Layer \Rightarrow Turbulent boundary layers stay attached to bodies longer \Rightarrow Narrower wake! \Rightarrow Lower pressure drag!

8.2 Open-Channel Flow

Pipe/duct flow \Rightarrow closed, full, gas or liquid.
\Rightarrow Pressure - friction balance.

Open-channel flow \Rightarrow free-surface (river, flume, partially filled pipe, ...).
\Rightarrow Gravity - friction balance.

Open-channel flow is often turbulent with complex geometries (rivers, estuaries, streams!). Hence we will simplify the geometry (often an excellent approximation)!

\Rightarrow We will assume straight channels with simple geometries (prismatic channels) and steady state flow (in time).

Free-Surface \Rightarrow at constant pressure \Rightarrow atmospheric.

This helps analysis! But free-surface position is unknown! This hinders analysis!

There are entire books written on open-channel flow (French, Henderson, Chow, ...), we will only discuss the basics. For more details keep an eye open for CEE 3320 (may get a 4000 number) that will cover the details of unsteady open channel/river flow, sediment transport, and fundamental coastal engineering/water wave theory.
Consider the energy equation. At surface $P_1 = P_2 = P_{atm}$, therefore:

$$\frac{V_1^2}{2g} + z_1 = \frac{V_2^2}{2g} + z_2 + h_f$$

From our pipe flow work we should immediately suspect that it is a reasonably good approximation to consider:

$$h_f \approx f \frac{L}{D_h} \frac{V_{avg}^2}{2g}$$

where $L = x_2 - x_1$ and $D_h = 4A/P$ as previously defined. Note $P$ is the wetted perimeter hence only the distance along the wetted sides of the channel.

Now, we can relate $V_1$ to $V_2$ from conservation of mass since $Q_1 = Q_2 \Rightarrow V_2 = V_1 \frac{A_1}{A_2}$.

We define an appropriate local Reynolds number as $Re_{D_h} = U D_h / \nu$. If $Re_{D_h} \gtrsim 10^5$ the flow is turbulent and most are (classic exception is sheet flow of paved surfaces).
8.4 Flow Classification by Variation of Depth with Distance Along Channel

- **Uniform Flow** constant depth $\Rightarrow \frac{dy}{dx} = 0$.
- **Varied Flow** non-constant depth $\Rightarrow \frac{dy}{dx} \neq 0$.
  - **Gradually Varied Flow** $\Rightarrow \frac{dy}{dx} \ll 1$.
  - **Rapidly Varied Flow** $\Rightarrow \frac{dy}{dx} \sim 1$.

A picture:

8.5 Uniform Flow

In uniform flow $y_1 = y_2$, $V_1 = V_2 = V$, therefore the energy equation becomes:

$$z_1 - z_2 = S_0 L = h_f$$

Flow is essentially fully developed, therefore we can apply Darcy-Weissbach relations.

$$h_f = f \frac{L}{D_h} \frac{V_{avg}^2}{2g}$$

In open channel flow it is more common to work with the:

$$\text{hydraulic radius} = \frac{A}{P} = \frac{D_h}{4} = R_h$$
where again \((P)\) is the length of the wetted perimeter. Combining the above two equations we have:

\[
S_0 L = f \frac{L}{D_h} \frac{V_{avg}^2}{2g} \quad \Rightarrow \quad V = \left( \frac{8g}{f} \right)^{1/2} (R_h S_0)^{1/2}
\]

### 8.6 Chézy Formulas

Chézy defined the coefficient

\[
C = \left( \frac{8g}{f} \right)^{1/2}
\]

(now called the Chézy coefficient) and found that it varies by a factor of 3. Therefore

\[
V = C (R_h S_0)^{1/2}
\]

and

\[
Q = C A (R_h S_0)^{1/2}
\]

Manning did field tests and found

\[
C = \left( \frac{8g}{f} \right)^{1/2} \approx \frac{\alpha}{n} \frac{R_h^{1/6}}{n}
\]

where \(n\) is known as Manning’s \(n\) and is a roughness coefficient and \(\alpha\) is a dimensional constant that varies with systems of units (this is not a homogeneous equation, remember?!). For SI \(\alpha = 1\), for BG \(\alpha = 1.486\). It is left as an exercise for the student to find the units and verify the conversion.

### 8.7 Manning’s Equation

Substituting Manning’s result into the Chézy formulas we have the celebrated Manning’s equation:

\[
V \approx \frac{\alpha}{n} R_h^{2/3} S_0^{1/2} \quad \text{and} \quad Q \approx \frac{\alpha}{n} A R_h^{2/3} S_0^{1/2}
\]

\(n\) varies by a factor of 15 and is tabulated in your text and more extensively elsewhere.
8.8 Examples – Fall Creek Flow

$S_0 \approx 0.002$, $n = 0.035$ from table 10.1 in text, assuming somewhere between clean and straight and sluggish with deep pools values, $b = 50'$, $d = 1.01'$ from USGS gage on web, 11/18/2012 @ 17:45 EST.

What is $Q$?

\[ Q = \frac{\alpha}{n} AR_{h}^{2/3} S_{0}^{1/2} \]

\[ R_{h} = \frac{A}{P} = \frac{bd}{2d + b} \approx d \quad \text{if} \quad b \gg d \]

therefore

\[ Q = \frac{\alpha}{n} bd^{5/3} S_{0}^{1/2} = \frac{1.486 \text{ ft}^{1/3}/s}{0.035} (50 \text{ ft})(1.01 \text{ ft})^{5/3} \sqrt{0.002} \]

$\Rightarrow \quad Q = 97$ CFS

Actual value from calibrated flow gage was 73 CFS. Just using ball park estimates we were in the right range. To be more accurate we would need a survey of the river slope and the wetted perimeter, perhaps also a bit less than our 50' estimate. We could get these more accurately from a USGS topographic map but to be truly accurate we would send a survey team out to measure directly in the field.

**Second example - Given $Q$, find $d$**

If $d \ll b$ straightforward – the equation is explicit and we just solve for $d$ directly by substitution into Manning’s equation.

However, let’s not make this assumption and let’s consider a trapezoidal shaped channel with the geometry shown below and $Q = 97$ CFS, $S_0 = 0.002$, and $n = 0.035$:

\[ A = \left( \frac{20 + b}{2} \right) d \]
Using Pythagoras' theorem we can write

\[ P = 20 + 2\sqrt{\left(\frac{b - 20}{2}\right)^2 + d^2} \]

and

\[ b = 20 + 2d \frac{15}{1.1} \]

Therefore

\[ A = \left(20 + d \frac{15}{1.1}\right)d \]

\[ P = 20 + 2\sqrt{\left(d \frac{15}{1.1}\right)^2 + d^2} \]

Therefore:

\[ Q = \frac{\alpha A^{5/3}}{n P^{2/3} S_0^{1/2}} = \frac{1.486 \text{ ft}^{1/3}/s}{0.035} \left[\frac{(20 + d \frac{15}{1.1})d}{20 + 2\sqrt{(d \frac{15}{1.1})^2 + d^2}}\right]^{2/3} \sqrt{0.002} = 97 \text{ ft}^3/s \]

Ouch - what to do? We can solve this iteratively by guessing a \( d \) and then fiddling until we find \( Q = 169 \text{ CFS} \). Alternatively, we can use an equation solver to find the result. I used the function fzero in Matlab which is a nonlinear root finder and found the zeroes to the expression:

\[ \text{residual} = Q - \frac{\alpha A^{5/3}}{n P^{2/3} S_0^{1/2}} \]

with an initial guess of \( d=3 \text{ ft} \). The result is \( d = 1.38 \text{ ft} \).
8.9 Energy Equation - Revisited

Our last work with the energy equation left us with:

$$\frac{V_1^2}{2g} + z_1 = \frac{V_2^2}{2g} + z_2 + h_L$$

But we can write $h_L = S_f L$. Defining $\xi = z - y$ we have $\xi_1 = \xi_2 + S_0 L$ and

$$\frac{V_1^2}{2g} + y_1 + S_0 L = \frac{V_2^2}{2g} + y_2 + S_f L$$

hence

$$\frac{V_1^2}{2g} + y_1 = \frac{V_2^2}{2g} + y_2 + (S_f - S_0)L$$

8.9.1 Specific Energy

Define $E = y + \frac{V^2}{2g}$ which is known as the specific energy.

The specific energy is seen to be the height of the EGL above the bed. Let $q = Q/b = V y$ where $b$ is the width of the channel. We can express the specific energy as

$$E = y + \frac{q^2}{2gy^2}$$

We see that this curve is a cubic in $y$ – let’s look at the shape of this plot:

Let’s look at the minimum in $E$. We can find its location (which we will denote $y_c$, the critical depth) by solving

$$\frac{\partial E}{\partial y} = 0 = 1 - \frac{2q^2}{2gy_c^3} \quad \Rightarrow \quad y_c = \left(\frac{q^2}{g}\right)^{1/3} = \left(\frac{Q^2}{b^2g}\right)^{1/3}$$
Now we can define the minimum in the specific energy curve ($E_{\text{min}}$)

$$E_{\text{min}} = E(y_c) = \left(\frac{q^2}{g}\right)^{1/3} + \frac{q^2}{g} = \frac{q^{2/3}}{g^{1/3}} + \frac{q^{2/3}}{2g^{1/3}} = \frac{3q^{2/3}}{2g^{1/3}} = \frac{3}{2}y_c$$

Defining $q = V_y = V_c y_c$ we can substitute into our definition of $y_c$

$$y_c^2 = \frac{q^2}{g} = \frac{V_c^2 y_c^2}{g} \Rightarrow V_c^2 = g y_c \Rightarrow V_c = \sqrt{g y_c}$$

Recall from Lab #3 and dimensional analysis that

$$\text{Fr} = \frac{V}{\sqrt{gy}} \Rightarrow \text{Fr}_c = \frac{V_c}{\sqrt{g y_c}} = \frac{\sqrt{g y_c}}{\sqrt{g y_c}} = 1$$

Hence at the inflection point in the specific energy curve we have

$$Fr = 1, \quad y = y_c, \quad E = E_{\text{min}} = \frac{3}{2}y_c$$

thus the inflection point separates supercritical flow from sub-critical flow (recall Lab #3 for the definitions of super- and sub-critical flow). The upper portion of the specific energy curve is known as the subcritical branch (deeper, slower flows at a given energy) while the lower branch is known as the supercritical branch) shallower, faster flows at a given energy).