2.3 Review

- Surface tension: $\Delta P$ is inversely related to $R$ - the surface curvature (e.g., for bubble we found $\Delta P = 2\pi/R$).

- Contact angle: defined as the angle between the tangent to the free surface directed from the solid-liquid-gas interface to the solid boundary through the liquid.

2.3.1 Incompressible Fluids

\[ dP = -\gamma \, dz \Rightarrow \int_{P_1}^{P_2} dP = -\int_{z_1}^{z_2} \gamma \, dz \Rightarrow P_2 - P_1 = -\int_{z_1}^{z_2} \gamma \, dz \]

Now let's consider particular solutions:

**Constant $\gamma$**

It is usually reasonable to assume that all liquids essentially have a constant $\gamma$, certainly true for all fluids at constant temperature and pressure. Under this condition we have

\[ P_2 - P_1 = -\gamma \int_{z_1}^{z_2} dz = -\gamma (z_2 - z_1) = -\gamma h \]

Therefore

\[ P_1 = \gamma h + P_2 \quad \text{or} \quad h = \frac{P_1 - P_2}{\gamma} \]

where we refer to $h$ as the *pressure head* as it is a pressure measured in units of length.

**Liquids with a Free Surface**

Examples include lakes, rivers, oceans, reservoirs, your pint glass of water, ...
The surface is the datum (at this point the pressure is the pressure of the atmosphere or surrounding gas). Therefore we have $z = 0, P = P_{\text{atm}}$.

$$P_1 = \gamma h + P_{\text{atm}} \quad \text{but} \quad h = z_2 - z_1 = -z_1 \quad \Rightarrow \quad P = P_{\text{atm}} - \gamma z$$

where $z < 0$ in liquid (water) and $z > 0$ in gas (atmosphere).

**Compressible Gases**

$$\frac{dP}{dz} = -\gamma = -\rho g = -\frac{P}{R\Theta}g$$

How do we solve this? \Rightarrow separation of variables!

$$\int_{P_1}^{P_2} \frac{dp}{P} = -\int_{z_1}^{z_2} \frac{g\,dz}{R\Theta}$$

we assume $g$ is a constant but that $\Theta$ may vary (e.g., the atmosphere). Then we can write the above

$$\ln \frac{P_2}{P_1} = -\frac{g}{R} \int_{z_1}^{z_2} \frac{dz}{\Theta}$$

If we assume that the temperature is constant within the elevation range of interest, an assumption known as isothermal conditions (i.e., $\Theta = \Theta_0 = \text{a constant}$), we can write

$$\ln \frac{P_2}{P_1} = -\frac{g}{R\Theta_0}(z_2 - z_1) \quad \text{or} \quad P_2 = P_1 \exp \left(-\frac{g(z_2 - z_1)}{R\Theta_0}\right)$$

### 2.3.2 The Standard Atmosphere

In the troposphere (first 11.0 km of atmosphere) the temperature drops linearly with elevation. The rate of temperature drop is known as the lapse rate and is given by $\beta = 0.00650 \, \text{K/m (6.5 °C/km or 3.6 °F/1000′)}$. Hence in the troposphere we write

$$\Theta = \Theta_a - \beta z$$

where $\Theta_a$ is the temperature at sea level which for the standard atmosphere is taken to be 15°C. We can use the expression we found for compressible gasses (recall we allowed
for temperature variation!) and integrate to find

\[ P = P_a \left( 1 - \frac{\beta z}{\Theta_a} \right) \frac{a}{R} \beta \]

This is the expression for the pressure in the troposphere.

2.4 Measurement of Pressure

*Absolute pressure* is the pressure relative to a perfect vacuum, hence it is always a positive (or zero) value.

![Figure 2.1: U.S. standard atmosphere to 20.1 km](image-url)
Gage pressure is the pressure relative to the local atmospheric value

\[ P > P_a \quad \Rightarrow \quad P_{gage} = P - P_a > 0 \]

\[ P < P_a \quad \Rightarrow \quad P_{gage} = P - P_a < 0 \quad P_{vacuum} = P_{suction} = P_a - P > 0 \]

Units of pressure are force/area hence pounds per square inch (psi), pounds per square foot (psf), newtons per square meter (N/m²=pascal (Pa)).

We will take all pressures, unless otherwise noted, to be gage.

Example – Liquids with a Free Surface

2.4.1 Barometer

\[ P_{atm} - P_{vapor} = \gamma h \quad \Rightarrow \quad P_{atm} = \gamma h + P_{vapor} \]

For mercury \( P_{vapor} \sim 0 \quad \Rightarrow \quad P_{atm} = \gamma h. \)

Typical values of pressure (from the U.S. standard atmosphere at sea level) are: 14.7 psia, 101 kPa, or, 760. mm Hg, 29.9 in Hg, 33.9 ft water (recall \( \gamma_{water} = 62.4 \text{ lbs/ft}^3 \) and S.G.\textsubscript{mercury} = 13.6).
2.4.2 Manometry

A manometer is a vertical or inclined tube used to measure pressure. There are three fundamental types of manometers: Piezometer tube, U-tube, and inclined tube.

Manometer analysis is straightforward hydrostatics - the key is to keep track of the signs of the pressure terms! Consider the following U-tube manometer:

\[ P_2 - P_1 = -\gamma(z_2 - z_1) \]

Now we track the pressure from A to 1 and from 1 to 2. A simple rule, based on our understanding that as we move down in a static fluid the pressure increases, is:

\[ P_{\text{down}} = P_{\text{up}} + \gamma|\Delta z| \]

This obviates the need to keep track of signs of directions in the vertical. Hence for our U-tube problem we have

\[ P_A + \gamma_1|z_A - z_1| - \gamma_2|z_2 - z_1| = P_{\text{atm}} \]

and we solve for \( P_A \).
Example - Inclined Manometer