3.6 Review

- **Volume flow rate** $(Q)$ & **Mass Flow Rate** $(\dot{m} = \rho Q)$
- **Flux** – the amount of stuff crossing a unit area in a unit of time
- **Reynolds Transport Theorem**
  \[ \frac{dB_{\text{sys}}}{dt} = \frac{dB_{\text{CV}}}{dt} + \rho_{\text{out}}\beta_{\text{out}}Q_{\text{out}} - \rho_{\text{in}}\beta_{\text{in}}Q_{\text{in}} \]

3.7 Conservation of Linear Momentum

Newton’s second law for a fluid. It is a Lagrangian conservation law, as we have already discussed, hence we write

*Time-rate-of-change of the linear momentum of the system = Sum of the external forces*

or

\[ \frac{d}{dt} \int_{\text{sys}} \vec{v} \rho \, d\mathcal{V} = \sum \vec{F}_{\text{sys}} \]

which is true in an *inertial* (non-accelerating) reference frame.

To derive the conservation of linear momentum we substitute momentum $(m\vec{v})$ into the Reynolds Transport Theorem, which for $B = m\vec{v}$ yields $\beta = \vec{v}$, and gives us

\[ \frac{d}{dt} \int_{\text{sys}} \vec{v} \rho \, d\mathcal{V} = \frac{d}{dt} \int_{\text{C.V.}} \vec{v} \rho \, d\mathcal{V} + \int_{\text{C.S.}} \vec{v} \rho (\vec{v} \cdot \vec{n}) \, dA = \sum \vec{F}_{\text{sys}} \]

But at a particular instant in time $\sum \vec{F}_{\text{sys}} = \sum \vec{F}_{\text{C.V.}}$ and we write

\[ \frac{d}{dt} \int_{\text{C.V.}} \vec{v} \rho \, d\mathcal{V} + \int_{\text{C.S.}} \vec{v} \rho (\vec{v} \cdot \vec{n}) \, dA = \sum \vec{F}_{\text{C.V.}} \]

What is $\sum \vec{F}_{\text{C.V.}}$? It is the sum of surface forces and body forces.

- surface forces: Pressure acting on an area, shear stress acting on an area, ...
- body forces: Gravity, electromagnetic, ...
Example of Determining $\sum \vec{F}_{C.V.}$

Consider the nozzle:

$$\sum \vec{F}_{C.V.} = 707 \text{ lbs}$$

### 3.7.1 Vector Nature of Equation and 1-D Form

Keep in mind that this is a vector equation! The surface integrals lead to momentum fluxes ($\dot{m}\vec{v}$) across the surfaces. We hold to our established sign convention set by the positive outward facing normal vector to surface areas and find that the momentum fluxes out of the volume are positive while the momentum fluxes into the volume are negative.

- $\vec{v} \cdot \vec{n} > 0$ outward momentum flux
- $\vec{v} \cdot \vec{n} < 0$ inward momentum flux

regardless of the sign of $\vec{v}$. The sign of $\vec{v}$ establishes the sign of the momentum and hence you can have an inwardly directed negative momentum flux which yields, in 1-D, $\dot{m}\vec{v} > 0$.

If the flow is one-dimensional then $\vec{v}$ and $\rho$ are uniform at all inlets and outlets. Therefore we can write

$$\int_{C.S.} \vec{v}\rho (\vec{v} \cdot \vec{n}) \, dA = \vec{v}_i (\rho_i v_i A_i) = \pm \dot{m}\vec{v}_i$$
where the ± arises from the dot product and determines whether the momentum flux is inward or outward. Hence our complete equation for a 1-D system is:

$$\frac{d}{dt} \int_{C.V.} \vec{v} \rho d\vec{v} + \sum (\dot{m}_i \vec{v}_i)_{out} - \sum (\dot{m}_i \vec{v}_i)_{in} = \sum \vec{F}_{C.V.}$$

Example – Conservation of Linear Momentum

What are $F_x$ and $F_z$ on the vane?

$$F_x = \dot{m}V (\cos \theta - 1) = \rho A V^2 (\cos \theta - 1), \quad F_z = \dot{m}V \sin \theta = \rho A V^2 \sin \theta$$

How does $\vec{F}$ behave as a function of $\theta$?

$$|\vec{F}| = 2\dot{m}V \sin \frac{\theta}{2}$$
3.8 Further discussion of Conservation of Linear Momentum and Control Volumes

Let’s revisit our nozzle example worked previously but this time let’s solve for $F_x$, the reaction force required to support the nozzle

\[
\frac{d}{dt} = 0 \quad \Rightarrow \quad -\dot{m}_1 u_1 + \dot{m}_2 u_2 = -F_x + (P_0 + P_{atm})A_1 - P_{atm}A_1
\]

But $\dot{m}_1 = \dot{m}_2 = \dot{m}$, therefore

\[
F_x = \dot{m}(u_1 - u_2) + P_0 A_1
\]

What about forces on the nozzle only?

Let $R_x$ be the interaction between the fluid and the inside surface of the nozzle

\[
F_x = (A_2 - A_1)P_{atm} + R_x
\]
What about forces on the fluid only?

\[ R_x = \dot{m}(u_1 - u_2) + (P_0 + P_{atm})A_1 - P_{atm}A_2 \]

Putting our above two equations together we should get the same result!

And we do! The determined forces depend strongly on the choice of control volume!

### 3.8.1 Variations From Uniform Flow - the Momentum Flux Correction Factor

As we have discussed we frequently assume that a flow is 1-D while we know in actuality it is not. Often this is an excellent assumption but sometimes the assumption is not as good and we may wish to correct for the effects of the dependence of the velocity on position. The terms that are effected are clearly the nonlinear terms so in the linear momentum equation the flux term is affected. If we wish to use the average velocity, \( \langle V \rangle \), as representative of a 1-D velocity equivalent to the 2-D velocity then we have

\[
\int_{CS} \vec{v}\rho(\vec{v} \cdot \vec{n}) \, dA = \dot{m}_{out}\beta_{out} \langle V \rangle_{out} - \dot{m}_{in}\beta_{in} \langle V \rangle_{in}
\]

where \( \beta \geq 1 \) is known as the momentum flux correction factor and it accounts for the effect of the non-uniform velocity profile on the surface momentum flux. The definition of the mean velocity is

\[
\langle V \rangle = \frac{\int \rho(\vec{v} \cdot \vec{n}) \, dA}{\rho A}
\]
which for incompressible flows with the velocity vector normal to the control surface reduces to simply $\langle V \rangle = Q/A$. Hence

$$\dot{m} \beta \langle V \rangle = \int_{CS} \vec{v} \rho (\vec{v} \cdot \vec{n}) \, dA$$

and

$$\beta = \frac{\int_{CS} \vec{v} \rho (\vec{v} \cdot \vec{n}) \, dA}{\dot{m} \langle V \rangle}$$

and if flow is normal to the entrance and exit planes of the C.V. (like it is in a pipe) then we can write:

$$\beta = \frac{\int_{CS} v^2 \, dA}{A \langle V \rangle^2}$$

### 3.8.2 Ex. - Momentum Flux Correction Factor

Let’s revisit our pipe entrance flow example from the mass conservation section to see how significant the momentum flux correction factor can be: