Chapter 4

Differential Analysis

4.1 Review

- Non-uniform velocity profile ⇒ kinetic factor, $\alpha$.

- Irrotational Flow & Bernoulli – The Bernoulli constant is a constant in irrotational flow regions.

- Stagnation pressure – the sum of the static pressure and the dynamic pressure.

  $$P_s = P + \rho \frac{v^2}{2}$$

- Pitot-static tube – $V = \sqrt{2\frac{\Delta P}{\rho}} = \sqrt{2\frac{\Delta H}{\rho}} = \sqrt{2g\Delta H}$, a particular case of the Bernoulli equation, where $\Delta P$ or $\Delta H$ is the measured differential pressure or head, respectively, between the stagnation and static pressures.

- Torricelli’s Law – $V = \sqrt{2gh}$, a particular case of the Bernoulli equation.
4.2 Acceleration Revisited - The Eulerian Perspective

We define the acceleration of a particle or fluid parcel as

\[ a_x = \frac{du}{dt}, \quad a_y = \frac{dv}{dt}, \quad a_z = \frac{dw}{dt} \]

where

\[ u = \frac{dx}{dt}, \quad v = \frac{dy}{dt}, \quad w = \frac{dz}{dt} \]

clearly this is a Lagrangian description – we are describing the acceleration of a fluid parcel or particle which is moving with the flow. It is also apparent that the Eulerian and Lagrangian descriptions of velocity are identical, namely the time rate of change of position with respect to some point. Now what is the acceleration in an Eulerian reference frame?

We write the velocity as

\[ u = u(x, y, z, t), \quad v = v(x, y, z, t), \quad w = w(x, y, z, t) \quad \text{and} \quad \vec{u} = u\vec{i} + v\vec{j} + w\vec{k} \]

and by the chain rule we have

\[ a_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} + \vec{u} \cdot \nabla u = \frac{Du}{Dt} \]

where we recall that \( \frac{D}{Dt} \) is the \textit{substantial derivative}. Similarly

\[ a_y = \frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \frac{\partial v}{\partial t} + \vec{u} \cdot \nabla v = \frac{Dv}{Dt} \]

\[ a_z = \frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{\partial w}{\partial t} + \vec{u} \cdot \nabla w = \frac{Dw}{Dt} \]
Therefore,

\[ \vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k} = \frac{Du}{Dt} \vec{i} + \frac{Dv}{Dt} \vec{j} + \frac{Dw}{Dt} \vec{k} = \frac{D \vec{u}}{Dt} \]

4.2.1 The Substantial Derivative – an Example

If a hiker starts at 10,000 feet on a 20,000 foot hike at 8 am and is hiking to 14,000 feet and a reasonable model of the atmosphere is the standard atmosphere and a reasonable model of the diurnal effects of sun on temperature is \( \Theta = \Theta_{8am} + \alpha t \) where \( \alpha = 3^\circ C/hr. \) What velocity over ground must be sustained in order to experience a constant temperature?

0.8 mph.
4.3 Differential Analysis - Conservation of Mass

Consider the infinitesimal fixed control volume

\[ \int_{CV} \frac{\partial \rho}{\partial t} dV + \sum_i (\rho_i A_i v_i)_{out} - \sum_i (\rho_i A_i v_i)_{in} = 0 \]

Now, the control volume is infinitesimal so we can take \( \frac{\partial \rho}{\partial t} \) to be spatially constant.

Therefore

\[ \int_{CV} \frac{\partial \rho}{\partial t} dV = \frac{\partial \rho}{\partial t} dx dy dz \]

4.3.1 Flux Terms

\[ \left[ \rho u + \frac{\partial (\rho u)}{\partial x} \right] dy dz \]

If we extend our analysis to all three faces we find a total of three inflows and three outflows.

<table>
<thead>
<tr>
<th>Face</th>
<th>( \dot{m}_{in} )</th>
<th>( \dot{m}_{out} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( \rho u dy dz \left( \rho u + \frac{\partial (\rho u)}{\partial x} dx \right) ) dy dz</td>
<td></td>
</tr>
<tr>
<td>( y )</td>
<td>( \rho v dx dz \left( \rho v + \frac{\partial (\rho v)}{\partial y} dy \right) dx dz )</td>
<td></td>
</tr>
<tr>
<td>( z )</td>
<td>( \rho w dx dy \left( \rho w + \frac{\partial (\rho w)}{\partial z} dz \right) dx dy )</td>
<td></td>
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</tbody>
</table>

But \( \dot{m}_{in} > 0 \), \( \dot{m}_{out} < 0 \), therefore

\[ \frac{\partial \rho}{\partial t} dx dy dz + \frac{\partial (\rho u)}{\partial x} dx dy dz + \frac{\partial (\rho v)}{\partial y} dx dy dz + \frac{\partial (\rho w)}{\partial z} dx dy dz = 0 \]

Therefore

\[ \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0 \]
This is the conservation of mass equation written in differential form, also known as the continuity equation. We can also write it

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0
\]

### 4.3.2 Steady Flow

\[
\frac{\partial}{\partial t} = 0 \quad \Rightarrow \quad \nabla \cdot (\rho \vec{v}) = 0
\]

### 4.3.3 Steady, Incompressible Flow

\[
\frac{\partial \rho}{\partial t} = 0, \quad \rho = \text{constant}
\]

Therefore

\[
\rho (\nabla \cdot \vec{v}) = 0 \quad \text{and hence} \quad \nabla \cdot \vec{v} = 0 \quad \text{or} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

### 4.3.4 Example – water waves

Consider what is known as a linear deep water wave (a wave traveling in a fluid whose depth is sufficiently large that the wave does not know that there is any bottom). The equation for the \(x\)–component of the velocity of a wave traveling in the \(x\) direction is:

\[
u(x, z, t) = a \sigma \cos (kx - \sigma t) e^{kz}
\]

where \(k = 2\pi/\lambda\), \(\lambda\) is the wavelength of the water wave, \(\sigma = 2\pi/\lambda\), and \(T\) is the wave period. If the wave is 2-D (e.g., \(v = 0\) and \(\partial/\partial y = 0\)), and that water is incompressible,
what is the vertical velocity component?

\[ w(x, z, t) = a\sigma \sin (kx - \sigma t)e^{kz} \]

If you are intrigued by water waves consider taking CEE 4350 in spring 2013!

### 4.4 When is a Flow Incompressible?

Consider the 1-D steady continuity equation

\[ \frac{\partial (\rho u)}{\partial x} = 0 = \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} \]

Therefore, if a flow is to be considered incompressible

\[ \left| \frac{u}{\partial x} \right| \ll \left| \frac{\partial u}{\partial x} \right| \text{ or } \left| \frac{\delta \rho}{\rho} \right| \ll \left| \frac{\delta u}{u} \right| \]

Now, in section 1.9 of our text the speed of sound of gases, a secondary property of fluids, is discussed and equation 1.38 gives \( c^2 = \delta P/\delta \rho \) where \( c \) is the speed of sound. Hence

\[ \left| \frac{\delta P}{c^2 \rho} \right| \ll \left| \frac{\delta u}{u} \right| \]

Now we apply Bernoulli

\[ \frac{P_1}{\rho} + \frac{V_1^2}{2} = \frac{P_2}{\rho} + \frac{V_2^2}{2} \quad \Rightarrow \quad \rho \frac{V_1^2 - V_2^2}{2} = P_2 - P_1 \quad \text{let } V_1 = u, \ V_2 = u + \delta u \]

\[ \Rightarrow \quad \rho \frac{u^2 - (u + \delta u)^2}{2} = P_2 - P_1 \quad \Rightarrow \quad \delta P = -\rho u \delta u \]
Hence,
\[
\left| \frac{\rho u \delta u}{c^2 \delta u} \right| \lesssim \left| \frac{\delta u}{u} \right| \Rightarrow \left| \frac{u}{c^2} \right| \lesssim \left| \frac{1}{u} \right| \Rightarrow \text{Ma}^2 = \frac{u^2}{c^2} \lesssim 1
\]

Therefore if \( \text{Ma}^2 \lesssim 0.1 \) the flow can be assumed incompressible which is the same as \( \text{Ma} \lesssim 0.3 \)

Air at standard conditions \( c \approx 340 \text{ m/s} \) \( \Rightarrow \) therefore \( u < 100 \text{ m/s} \) or \( u < 220 \text{ mph} \).

For water we find \( c \approx 1480 \text{ m/s} \) \( \Rightarrow \) therefore \( u < 440 \text{ m/s} \) or \( u < 980 \text{ mph} \! \)

Therefore all liquids and most environmental gasses can be assumed incompressible. This is not to say that density differences are never important – keep in mind that the density of water is weekly a function of temperature, while the associated volume change of mass as a function of temperature is rarely significant the buoyancy force is and hence stratification (vertical gradients in density) in the environment is a critical aspect of environmental flows.