5.10 Review

Dimensional Analysis

Buckingham Pi Theorem: \[ k = n - r \]

where \( k \) is the number of dimensionless numbers (\( \Pi \)'s), \( n \) is the number of dimensional variables, and \( r \) is the number of physical dimensions.

Copepod Drag Force \[ C_f = g(Re), \] or \[ \frac{F}{\rho V^2 L^2} = g \left( \frac{\rho V L}{\mu} \right) \]

Found the above relationship two ways - by inspection and by a formal Buckingham Pi analysis.

Can interpret dimensionless numbers as the ratio of two dimensional numbers, such as the Reynolds number is the ratio of the viscous time scale to the advective time scale.

5.11 A Second Example - Drag on a Sphere and a Cylinder

As we have already found from our copepod example the drag on a submerged object is a function of Reynolds number. If we are to extend this analysis to general submerged objects we need to include the roughness (we really needed roughness for the copepod too if the appendages grow and/or present themselves with different length scale ratios with respect to the original characteristic length scale of the copepod's size), hence we have:

\[ C_D = g \left( \text{Re}, \frac{\epsilon}{D} \right) \]

First let’s consider a sphere. As already discussed the \( L^2 \) term is typically taken as an area. Hence for a sphere we take \( L^2 = \pi D^2 / 4 \) where \( D \) is the diameter of the sphere and
define
\[ C_D = \frac{F}{\frac{1}{2} \rho \pi V^2 D^2 / 4} = \frac{8F}{\pi \rho V^2 D^2} \]

Where \( F \) is the drag force and the factor of \( 1/2 \) is included for historical reasons (Because the dynamic pressure in the Bernoulli equation has a factor of \( \frac{1}{2} \) in it). Hence, if we build a model and enforce geometric similarity (so that the roughness is scaled appropriately as well as the diameter) then we have:

\[ \frac{8F_m}{\pi \rho_m V_m^2 D_m^2} = \frac{8F_p}{\pi \rho_p V_p^2 D_p^2} \Rightarrow F_p = F_m \frac{\rho_p}{\rho_m} \left( \frac{V_p}{V_m} \right)^2 \left( \frac{D_p}{D_m} \right)^2 \]

We see that to ensure Reynolds number similarity and hence total similarity for this problem, and assuming we run the model tests in the same fluid as the prototype, we require

\[ Re_m = Re_p = \frac{L_m U_m}{\nu} = \frac{L_p U_p}{\nu} \Rightarrow U_m = U_p \frac{L_p}{L_m} \]

Thus if the scale gets very small we will need to run the model at very high velocity - this may not be possible or may lead to compressibility effects (air flows) or cavitation (water flows).

Now let’s look at some data for two cases - \( C_D = g(Re) \) for both a cylinder and a sphere. For a cylinder we take the area to be the projected area of the cylinder which is simply a rectangle \( LD \) where \( L \) is the length of the cylinder and \( D \) is the diameter. Therefore we have

\[ C_D = \frac{2F}{\rho V^2 LD} \]

We will continue to define the \( Re \) based on the diameter (since we have two length scales we have to be explicit about which we are using). This is often written \( Re_D \). Finally, since we have more than one length scale in the problem, the ratio of these length scales is a dimensionless parameter that enters the problem hence we have

\[ C_D = h \left( Re_D, \frac{\epsilon}{D}, \frac{L}{D} \right) \]

Borrowing Fig 5.3 from our textbook (shown on next page) we see several interesting features.
• Clearly the functions $g$ and $h$ for spheres and cylinders are different.

• $C_D$ in each case drops with increasing Re and reaches a near constant value at $10^4 < Re < 10^5$ shortly after which a sudden drop occurs (the flow is transitioning from laminar to turbulent here).

• Clearly $\epsilon/D$ plays a role in determining $C_D$.

• $L/D$ is important but as $L \gg D$ the effect on $C_D$ asymptotes toward a constant value and the physics becomes independent of $L/D$. 
5.12 Summary of Dimensionless Parameters

Here are some of the most common dimensionless numbers that show up in fluid mechanics:

- **Re (Reynolds Number)** \( \frac{\rho VL}{\mu} = \frac{VL}{\nu} \) – The ratio of inertial forces to frictional forces.

- **Fr (Froude Number)** \( \frac{V}{\sqrt{gL}} \) – The ratio of inertial forces to gravitational forces.

- **We (Weber Number)** \( \frac{\rho V^2 L}{\sigma} \) – The ratio of inertial forces to surface tension forces.

- **Eu (Euler Number)** \( \frac{\Delta P}{\frac{1}{2} \rho V^2} \) – The ratio of pressure forces to inertial forces.

- **C_D (Drag Coefficient)** \( \frac{F_D}{\frac{1}{2} \rho V^2 A} \) – The ratio of drag forces to dynamic pressure forces.

- **St (Strouhal Number)** \( \frac{fL}{V} \) – The ratio of event frequency (often vortex shedding) and the advective frequency (inverse of the advective, or inertial, time scale).

- **Ma (Mach Number)** \( \frac{V}{c} \) – The square root of the ratio of inertial forces to compressibility effect forces – can be thought of as a Froude number for compressible flows.