8.9 Review

Open Channel Flow \Rightarrow Gravity – friction balance.

In general we take an energy equation approach.

Uniform Flow \Rightarrow \frac{\partial y}{\partial x} = 0 \quad \Delta z = S_0L = h_f

8.10 Examples – Fall Creek Flow

\( S_0 \approx 0.001, \quad n = 0.035 \) from table 10.1 in text, assuming somewhere between clean and straight and sluggish with deep pools values, \( b = 50', \quad d = 1.66' \) from USGS gage on web, 12/1/2013 @ 22:00 EST.

What is \( Q \)?

\[
Q = \frac{\alpha}{n} AR_h^{2/3} S_0^{1/2}
\]

\[
R_h = \frac{A}{P} = \frac{bd}{2d + b} \approx d \quad \text{if } b \gg d
\]

therefore

\[
Q = \frac{\alpha}{n} bd^{5/3} S_0^{1/2} = \frac{1.486 \text{ ft}^{1/3}/8}{0.035} (50 \text{ ft})(1.66 \text{ ft})^{5/3} \sqrt{0.001}
\]

\( \Rightarrow \quad Q = 282 \text{ CFS} \)

Actual value from calibrated flow gage was 249 CFS. Just using ball park estimates we were within 15% – pretty good! To be more accurate we would need a survey of the river slope and the wetted perimeter. We could get these more accurately from a USGS topographic map but to be truly accurate we would send a survey team out to measure directly in the field.
Second example - Given $Q$, find $d$

If $d \ll b$ straightforward – the equation is explicit and we just solve for $d$ directly by substitution into Manning’s equation.

However, let’s not make this assumption and let’s consider a trapezoidal shaped channel with the geometry shown below and $Q = 282$ CFS, $S_0 = 0.001$, and $n = 0.035$:

![Diagram of trapezoidal channel](image)

$$A = \left(\frac{20 + b}{2}\right)d$$

Using Pythagoras’ theorem we can write

$$\mathcal{P} = 20 + 2\sqrt{\left(\frac{b - 20}{2}\right)^2 + d^2}$$

and

$$b = 20 + 2d\frac{15}{1.1}$$

Therefore

$$A = \left(20 + d\frac{15}{1.1}\right)d$$

$$\mathcal{P} = 20 + 2\sqrt{\left(d\frac{15}{1.1}\right)^2 + d^2}$$

Therefore:

$$Q = \frac{\alpha A^{5/3}}{n \mathcal{P}^{2/3} S_0^{1/2}} = \frac{1.486 \text{ ft}^{1/3}/\text{s}}{0.035} \left[\left(20 + d\frac{15}{1.1}\right)d\right]^{5/3} \sqrt{0.001} = 282 \text{ ft}^3/\text{s}$$
Ouch - what to do? We can solve this iteratively by guessing a $d$ and then fiddling until we find $Q = 169$ CFS. Alternatively, we can use an equation solver to find the result. I used the function fzero in Matlab which is a nonlinear root finder and found the zeroes to the expression:

$$\text{residual} = Q - \frac{\alpha A^{5/3}}{n P^{2/3} S_0^{1/2}}$$

with an initial guess of $d=3$ ft. The result is $d = 2.68$ ft.

### 8.11 Energy Equation - Revisited

Our last work with the energy equation left us with:

$$\frac{V_1^2}{2g} + z_1 = \frac{V_2^2}{2g} + z_2 + h_L$$

But we can write $h_L = S_f L$. Defining $\xi = z - y$ we have $\xi_1 = \xi_2 + S_0 L$ and

$$\frac{V_1^2}{2g} + y_1 + S_0 L = \frac{V_2^2}{2g} + y_2 + S_f L$$

hence

$$\frac{V_1^2}{2g} + y_1 = \frac{V_2^2}{2g} + y_2 + (S_f - S_0) L$$

#### 8.11.1 Specific Energy

Define $E = y + \frac{V^2}{2g}$ which is known as the specific energy.
The specific energy is seen to be the height of the EGL above the bed. Let \( q = Q/b = V y \) where \( b \) is the width of the channel. We can express the specific energy as

\[
E = y + \frac{q^2}{2gy^2}
\]

We see that this curve is a cubic in \( y \) – let’s look at the shape of this plot:

Let’s look at the minimum in \( E \). We can find its location (which we will denote \( y_c \), the critical depth) by solving

\[
\frac{\partial E}{\partial y} = 0 = 1 - \frac{2q^2}{2gy_c^3} \quad \Rightarrow \quad y_c = \left( \frac{q^2}{g} \right)^{1/3} = \left( \frac{Q^2}{b^2g} \right)^{1/3}
\]

Now we can define the minimum in the specific energy curve \( E_{\text{min}} \)

\[
E_{\text{min}} = E(y_c) = \left( \frac{q^2}{g} \right)^{1/3} + \frac{q^2}{2g^{4/3}y_c^{2/3}} = \frac{q^{2/3}}{g^{1/3}} + \frac{q^{2/3}}{2g^{1/3}} = \frac{3q^{2/3}}{2g^{1/3}} \quad \Rightarrow \quad \frac{3}{2y_c}
\]

Defining \( q = V y = V_c y_c \) we can substitute into our definition of \( y_c \)

\[
y_c^3 = \frac{q^2}{g} = \frac{V_c^2 y_c^2}{g} \quad \Rightarrow \quad V_c^2 = g y_c \quad \Rightarrow \quad V_c = \sqrt{g y_c}
\]
Recall from Lab #3 and dimensional analysis that

\[ Fr = \frac{V}{\sqrt{gy}} \Rightarrow Fr_c = \frac{V_c}{\sqrt{gy_c}} = \frac{\sqrt{gg_c}}{\sqrt{gg_c}} = 1 \]

Hence at the inflection point in the specific energy curve we have

\[ Fr = 1, \quad y = y_c, \quad E = E_{\min} = \frac{3}{2}y_c \]

thus the inflection point separates supercritical flow from sub-critical flow (recall Lab #3 for the definitions of super- and sub-critical flow). The upper portion of the specific energy curve is known as the subcritical branch (deeper, slower flows at a given energy) while the lower branch is known as the supercritical branch) shallower, faster flows at a given energy).

### 8.12 Example - Flow Over a Sill (Bump)

Consider the following constant width channel:

If we take the bed to be horizontal \((S_0 = 0)\) away from the sill and the flow to be frictionless \((S_f = 0)\) then the Bernoulli form of the specific energy equation gives us (where \(q\) is a constant of the flow since the flow width is constant):

\[ E_1 = E_2 + \Delta h = y_2 + \frac{q^2}{2gy_2^3} + \Delta h \]

Rearranging to be a polynomial in \(y_2^3\) we find:

\[ y_2^3 + (\Delta h - E_1)y_2^2 + \frac{q^2}{2g} = 0 \]
The coefficient terms to the polynomial are all known ($E_1$ is the left-hand side boundary condition and hence $y_1$ and $q$ and thus $E_1$ are all known for a well specified problem). Clearly this equation has three roots. It turns out that for $\Delta h$ not too large it has 3 real roots, one of which is negative (and hence is not physically possibly as a negative water depth does not make sense), leaving two real positive roots that are viable flow depths. Let’s consider the range of possibilities for this flow.

a) Initial Flow Subcritical Upstream and Downstream (e.g., $Fr_1 < 1$ and $Fr_3 < 1$)

b) Initial Flow Supercritical Upstream and Downstream (e.g., $Fr_1 > 1$ and $Fr_3 > 1$)

c) Initial Flow Subcritical Upstream and Supercritical Downstream (e.g., $Fr_1 < 1$ and $Fr_3 > 1$)
The specific energy plot for each flow looks like:
8.13 Example - Flow though a Contraction

Consider the following channel of constant bottom elevation with vertical banks:

As in our sill example we take the bed to be horizontal \((S_0 = 0)\) and the flow to be frictionless \((S_f = 0)\). However, now \(q\) is no longer constant throughout the flow as \(b = b(x) \Rightarrow q = Q/b = q(x)\). We consider the same three cases:

a) \(\text{Fr}_1 < 1\) and \(\text{Fr}_3 < 1\)

b) \(\text{Fr}_1 > 1\) and \(\text{Fr}_3 > 1\)
The specific energy plot for each flow looks like:

c) $Fr_1 < 1$ and $Fr_3 > 1$
8.14 Critical and Choked Flows

Looking at the previous two examples if the flow is in either of the two states (c) we see that the flow over the sill or through the throat (narrowest part of the channel) is critical. If the sill is raised at all or the throat is narrowed further (e.g., $q_2$ is raised further) the flow depth upstream ($y_1$) must increase in order to pass the flow. This is known as a choked flow. Note that the flow depth at the sill (throat) is still $y_2 = y_c$. Hence the flow is controlled at the sill (throat) and both the upstream and downstream flows are controlled by the sill (throat). This is possible as the flow upstream is subcritical hence information can propagate upstream from the control point to set the water depth upstream ($y_1$) and the flow is supersonic downstream hence information can propagate downstream from the control point to set the downstream water depth ($y_3$).

In fact the above explains why a fourth possible solution to the two example problems does not exist. Note we did not admit any solutions of the form $Fr_1 > 1$ and $Fr_3 < 1$ which would look like:

As this would require that a supersonic flow be controlled from downstream and a subsonic flow be controlled from upstream which is not possible.

How do supersonic flows transition to subsonic flows downstream? The hydraulic jump!
8.15 Rapidly Varied Flow

8.15.1 The Hydraulic Jump

As described a transition from super- to sub-critical flow.

- Extremely efficient energy dissipater
- Jump characteristics determined primarily by Fr

If we start with the conservation of energy and the continuity equation (our usual starting point so far) applied to a rectangular open-channel flow of width $b$ with a hydraulic jump we have:

Continuity $Q_1 = y_1 b V_1 = y_2 b V_2 = Q_2 \Rightarrow q_1 = q_2 = y_2 V_2 = y_1 V_1$

Energy conservation $E_1 = E_2 + h_L \Rightarrow y_1 + \frac{V_1^2}{2g} = y_2 + \frac{V_2^2}{2g} + h_L$

where we assume that the jump occurs over a short enough distance that $h_L$ only includes energy dissipated by the flow in the jump and not by shear stresses at the boundaries (the reasons will become clear in a moment).

- knowns – $y_1, V_1, g$
- unknowns – $y_2, V_2, h_L$
Therefore we have two equations and three unknowns – we need another equation \( \Rightarrow \) conservation of linear momentum. Our picture:

\[
\gamma \frac{y_1}{2} y_1 b - \gamma \frac{y_2}{2} y_2 b = \dot{m}_\text{out} - \dot{m}_\text{in} = \rho Q (V_2 - V_1) = \rho V_1 y_1 b (V_2 - V_1)
\]

Rearranging we have

\[
\frac{y_1^2}{2} - \frac{y_2^2}{2} = \frac{V_1 y_1}{g} (V_2 - V_1)
\]

Combing the above with continuity to eliminate \( V_2 \) and solving for the ratio \( \frac{y_2}{y_1} \) we arrive at

\[
\left( \frac{y_2}{y_1} \right)^2 + \frac{y_2}{y_1} - 2 \frac{V_1^2}{g y_1} = \left( \frac{y_2}{y_1} \right)^2 + \frac{y_2}{y_1} - 2 \text{Fr}_1^2
\]

Solving this quadratic equation we have

\[
\frac{y_2}{y_1} = \frac{-1 \pm \sqrt{1 + 8 \text{Fr}_1^2}}{2} \quad \Rightarrow \quad \frac{y_2}{y_1} = \frac{-1 + \sqrt{1 + 8 \text{Fr}_1^2}}{2}
\]

since clearly \( \frac{y_2}{y_1} > 0 \) and \( \text{Fr} > 1 \).

This result can be combined with the energy and continuity equations to solve for the head loss \( h_L \) through the jump.

What does a hydraulic jump look like on a specific energy diagram?
8.15.2 The Broad-Crested Weir

Consider the following flow:

This is known as the broad-crested weir which is characterized by:
• Sufficiently short that energy loss due to channel friction is negligible \( \Rightarrow h_L = 0 \) \( \Rightarrow \) Bernoulli’s equation.

• Sufficiently long horizontal section that hydrostatic flow is a reasonable approximation \( \Rightarrow \) pressure over horizontal section is hydrostatic.

Therefore our starting point is the Bernoulli equation

\[
E_1 = E_2 + h_w \quad \Rightarrow \quad \frac{V_1^2}{2g} + H + h_w = \frac{V_2^2}{2g} + h_w + y_2
\]

Aha! Since the flow upstream is subcritical and there is a region where \( \frac{dy}{dx} \sim 1 \) just upstream of the weir the flow over the horizontal section must be a control and hence \( y_2 = y_c \) and \( V_2^2 = V_c^2 = gy_c \). Therefore we have

\[
\frac{V_1^2}{2g} + H = \frac{y_c}{2} + y_c = \frac{3}{2}y_c
\]

Solving for \( y_c \) we have

\[
y_c = \frac{V_1^2}{3g} + \frac{2}{3}H \approx \frac{2}{3}H \quad \text{if} \quad \frac{V_1^2}{2g} \ll H
\]

Therefore a reasonable approximation of the flow rate over a weir is

\[
Q = by_c V_c = by_c \sqrt{gy_c} = b \sqrt{gy_c^{3/2}} = b \sqrt{gy^{3/2}} = \left(\frac{2}{3} \right)^{3/2} b \sqrt{gH^3}
\]

Now the reality is while the above is reasonable, there are energy losses and often broad-crested weirs are used as flow discharge measurement devices. Hence experimental calibration is often preferred and a *weir discharge coefficient* – \( C_d \) is experimentally determined. I.e.,

\[
Q = C_d b \sqrt{gH^3}
\]

The literature is full of different formulations such as equations 10.57 and 10.58 in your textbook which yields a maximal \( C_d \) of about 0.54.