Chapter 3

Control Volume Analysis

3.1 Review

- Hydrostatic force on curved surfaces ⇒ control volume approach
- Line of Action
- Buoyancy – weight of fluid displaced by a body.

3.2 Systems & Control Volumes

A system is a particular collection of matter separated from everything external by imaginary or real closed boundaries.
A systems’ mass is conserved. This is so fundamental in solid mechanics that it is not often written down:

\[ m = \text{const} \quad \text{or} \quad \frac{dm}{dt} = 0 \]

which is a fundamental law of mechanics, the conservation of mass. A system based analysis of fluid flows involves the Lagrangian reference frame, however, the system is moving and deforming—therefore it is hard to track its boundary! Consider turbulent flows, in these flows it is hard to even identify the boundaries!

A Control Volume (CV) is a defined volume in space, always identifiable, that may move and or deform but in an independent manner from the flow field. Mass, momentum and energy flows across the boundaries, known as control surfaces, that demarcate the control volume. This is an Eulerian type view (fixed size, location and shape, perhaps moving but at arbitrary velocity with respect to the flow itself).

We recognize that Newton’s second law, that unbalanced forces lead to accelerations

\[ \vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt} = \frac{d}{dt}(m\vec{V}) \]

applies to systems (since it applies to a unique collection of mass, which are what was analyzed in solid mechanics) but in most cases we will not wish to follow a fluid system (e.g., a slug of water as it goes through, and ultimately out of, a hose) but instead will choose to define a control volume and describe the flow from the perspective of this fixed volume (e.g., we analyze the volume contained within the hose and we have flow in and out of this volume). Newton’s second law in fluid mechanics is known as the Conservation of Linear Momentum and it clearly has three component equations in the three Cartesian directions.

A third law of mechanics arises from unbalanced moments on systems

\[ \vec{M} = \frac{d\vec{H}}{dt} \quad \text{where} \quad \vec{H} = \sum \left( \vec{r} \times \vec{V} \right) \delta m \]

which basically says that unbalanced moments about the center of mass lead to rotation.

In fluid mechanics this will lead to the the Conservation of Angular Momentum.
Finally, if heat $\delta Q$, is added to a system or work $\delta W$ is done by the system on the surroundings, the system energy must change according to the first law of thermodynamics

$$\delta Q - \delta W = dE \quad \text{or} \quad \dot{Q} - \dot{W} = \frac{dE}{dt}$$

which in fluid mechanics will lead to the the Conservation of Energy.

### 3.2.1 Volume and Mass Flow Rate

In control volume analysis we expect flows across surfaces.

What is the volume of flow across $S$ per unit time?

$$d\dot{V} = \delta t \, \vec{v} \cdot dA = v \delta t \cos \theta \, dA = (\vec{v} \cdot \vec{n}) \, dA \, \delta t$$

Now, we define $Q=$volume/time=volume flow rate. Then:

$$Q = \int_S \frac{d\dot{V}}{\delta t} = \int_S (\vec{v} \cdot \vec{n}) \, dA$$

Now, we define the normal vector $\vec{n}$ to be positive in the outward direction then

$$\vec{v} \cdot \vec{n} > 0 \quad \text{outflow}$$

$$\vec{v} \cdot \vec{n} < 0 \quad \text{inflow}$$

Further, if we multiply by the density, $\rho$, we have the mass flow rate

$$\dot{m} = \int_S \rho (\vec{v} \cdot \vec{n}) \, dA$$

If $\rho$ is a constant then

$$\dot{m} = \rho Q$$

If we can assume the flow is one-dimensional then we can write

$$\dot{m} = \rho Q = \rho AV \quad \text{where } V = \text{the average velocity across } A$$
3.2.2 Volume & Mass Flow Rate – an Example

Consider the pipeline reducer:

If the pipe fluid is water with a mass flow rate of $\dot{m} = 300 \text{ kg/s}$, what is $Q_1$, $Q_2$, $V_1$, $V_2$?

$Q_1 = Q_2 = Q = 0.3 \text{ m}^3/\text{s}$, $V_1 = 4.24 \text{ m/s}$, $V_2 = 9.55 \text{ m/s}$

3.2.3 Flow Rates Per Unit Area – the Flux

Consider the flow rate across any surface per unit surface area – this is known as the flux. Thus if we consider the volumetric flow rate per unit area this has the dimensions of:

$$\vec{q} = \frac{[L^3]}{[T \cdot L^2]} = \frac{[L]}{[T]} = \text{a velocity!}$$

which in this case is the velocity normal to the surface. If we consider the mass flux (mass flow rate per unit area) this has the dimensions of:

$$\vec{q}_m = \frac{[M]}{[T \cdot L^2]}$$

Note that flux is a vector quantity (in the direction normal to a surface). Often we speak of the total or net flux, which in the case of the volume and mass flow rates is just the volume and mass flow rates themselves.