3.2 Review

- Line of Action – the direction of the resultant force vector, $\vec{F}_r$, which passes through the center of pressure, which we know is located below the centroid unless the surface is horizontal.

- Stability – a function of the moment arm between the buoyant force and the weight of the object, the two of which form a couple. Note buoyant force occurs at the center of the volume of the displaced fluid. Weight acts through the center of gravity.

- Buoyancy – weight of fluid displaced by a body.

3.3 The Reynolds Transport Theorem

We need a way to connect our systems approach to a control volume approach, let’s track a one-dimensional fixed system and control volume in the same flow and see if we can connect the two. Consider the following pipe expansion flow:

At $t = t$, the system = control volume
At $t = t_0 + dt$, the system = (control volume - I)+II

Let $B$ be any fluid property (e.g., mass, velocity, energy ...), known as an extensive
quantity) then we can define

\[ \beta = \frac{B}{m} \]

as the amount of \( B \) per unit mass, this is known as an *intensive* quantity.

The total amount of \( B \) in the control volume can be found as

\[ B_{CV} = \int_{CV} \beta \rho \, d\mathcal{V} \]

where \( \rho \, d\mathcal{V} \) is seen to be an infinitesimal mass, \( \delta m \), within the CV.

Now, we are interested in the time-rate-of-change of \( B \) within the control volume so we have

\[ \frac{dB_{CV}}{dt} = \frac{B_{CV}(t + dt) - B_{CV}(t)}{dt} \]

\[ = \frac{B_2(t + dt) - (\beta \rho \, d\mathcal{V})_{\text{out}} + (\beta \rho \, d\mathcal{V})_{\text{in}} - B_2(t)}{dt} \]

\[ = \frac{B_2(t + dt) - B_2(t)}{dt} - (\beta \rho AV)_{\text{out}} + (\beta \rho AV)_{\text{in}} \]

The first term on the right-hand-side is the rate-of-change of \( B \) within system 2 at the instant it occupies the control volume, which is the quantity we want to relate to the rate-of-change of \( B \) within the control volume itself, hence we re-write the above:

\[ \frac{dB_{sys}}{dt} = \frac{dB_{CV}}{dt} - (\beta \rho AV)_{\text{out}} + (\beta \rho AV)_{\text{in}} \]

or, invoking our expression for \( B_{CV} \) above, we have

\[ \frac{dB_{sys}}{dt} = \frac{d}{dt} \left( \int_{CV} \beta \rho \, d\mathcal{V} \right) + (\beta \rho AV)_{\text{out}} - (\beta \rho AV)_{\text{in}} \]

which we can write in a simplified form

\[ \frac{dB_{sys}}{dt} = \frac{dB_{CV}}{dt} + \dot{B}_{\text{out}} - \dot{B}_{\text{in}} \]

For our simple 1-D constant control volume system we can write

\[ \frac{dB_{sys}}{dt} = \frac{dB_{CV}}{dt} + \rho_{\text{out}} \beta_{\text{out}} Q_{\text{out}} - \rho_{\text{in}} \beta_{\text{in}} Q_{\text{in}} \]

This is the *Reynolds Transport Theorem* for a fixed control volume one input, one output system.
3.3.1 Generalized Reynolds Transport System

We can generalize our above result to arbitrary shaped control volumes as

\[
\frac{dB_{sys}}{dt} = \frac{d}{dt} \left( \int_{CV} \beta \rho \, dV \right) + \int_{CS} \rho \beta (\vec{v} \cdot \vec{n}) \, dA
\]

And if the flow is steady we can write the time derivative of the control volume term as

\[
\frac{dB_{sys}}{dt} = \left( \int_{CV} \frac{\partial}{\partial t} (\beta \rho) \, dV \right) + \int_{CS} \rho \beta (\vec{v} \cdot \vec{n}) \, dA
\]

And if the control volume is moving at a velocity \( \vec{V}_{CV} \) an observer standing on the control volume would experience a relative velocity \( \vec{V}_r = \vec{v} - \vec{V}_{CV} \) and we can write our expression for the moving control volume as

\[
\frac{dB_{sys}}{dt} = \frac{d}{dt} \left( \int_{CV} \beta \rho \, dV \right) + \int_{CS} \rho \beta (\vec{V}_r \cdot \vec{n}) \, dA
\]

3.3.2 Simplifications

Consider:

Steady flow therefore \( \frac{\partial}{\partial t} = 0 \). Flow across a surface is a flux. Therefore the in/out terms in the above picture are flux terms.

The inlets and outlets are all 1-D, hence

\[
\int_S \rho \beta (\vec{v} \cdot \vec{n}) \, dA = \sum_{i=1}^{N} \rho_i \beta_i Q_i \left| \frac{\vec{v} \cdot \vec{n}}{\left| \vec{v} \cdot \vec{n} \right|} \right|
\]

\[
= \rho_1 \beta_1 V_1 A_1 + \rho_2 \beta_2 V_2 A_2 - \rho_3 \beta_3 V_3 A_3
\]
3.4 Conservation of Mass

We can derive the equation for conservation of mass, also known as the \textit{continuity equation}, by simply letting $B = m$ in the Reynolds Transport Theorem. Therefore, $\beta = 1$ and we have

$$\frac{dm_{sys}}{dt} = \frac{d}{dt} \int_{C.V.} \rho \, d\mathcal{V} + \int_{C.S.} \rho (\vec{v} \cdot \vec{n}) \, dA = 0$$

Now, if we have a fixed control volume then

$$\int_{C.V.} \frac{\partial \rho}{\partial t} \, d\mathcal{V} + \int_{C.S.} \rho (\vec{v} \cdot \vec{n}) \, dA = 0$$

And if the flow is at steady state:

$$\int_{C.S.} \rho (\vec{v} \cdot \vec{n}) \, dA = 0$$

If we have one-dimensional inlets and outlets

$$\sum (\rho_i A_i V_i)_{in} = \sum (\rho_i A_i V_i)_{out}$$

or

$$\sum (\dot{m}_i)_{in} = \sum (\dot{m}_i)_{out}$$

3.4.1 A Note on Average Velocity

The velocity $V$ in $\dot{m} = \rho Q = \rho A V$ represents the spatially averaged velocity across $A$ and hence really $\langle V \rangle$ where

$$\langle V \rangle = \frac{\int_A \rho (\vec{v} \cdot \vec{n}) \, dA}{\rho A}$$

Only if $\rho$ and $V$ are not functions of $A$ does $\langle V \rangle = V$ (i.e., only if the flow is truly one-dimensional).
Example

\[ \frac{dh}{dt} = \frac{\rho_1 A_1 V_1 + \rho_2 A_2 V_2}{\rho_w A} \]

### 3.4.2 Incompressible Flow, Fixed C.V.

For a fixed C.V. we have

\[ 0 = \int_{C.V.} \frac{\partial \rho}{\partial t} dV + \int_{C.S.} \rho (\vec{v} \cdot \vec{n}) dA \]

If the flow is incompressible then \( \frac{\partial \rho}{\partial t} = 0 \) and we have

\[ 0 = \int_{C.S.} \rho (\vec{v} \cdot \vec{n}) dA = \int_{C.S.} (\vec{v} \cdot \vec{n}) dA \]

Which states that whatever flows into the control volume must flow out since there is no change in storage in the control volume (the control volume is fixed in shape by assumption and the fluid is incompressible so we can not squeeze more in!). If the flow is one-dimensional then we can write

\[ \sum (V_i A_i)_{out} = \sum (V_i A_i)_{in} \text{ or } \sum (Q)_{out} = \sum (Q)_{in} \]

**Example - Pipe Entrance Flow**

\[ u = U_{max} \left( 1 - \frac{r^2}{R^2} \right) \]
If the inlet flow is uniform and denoted $U_0$, what is $U_{\text{max}}$?

$$U_{\text{max}} = 2U_0$$