3.15 Review

- Conservation of angular momentum - sign convention, CCW rotation > 0, CW rotation < 0
- 1-D form

$$\sum (\vec{r} \times \vec{F})_{ext} = \frac{\partial}{\partial t} \left[ \int_{CV} (\vec{r} \times \vec{v}) \rho \, dV \right] + \sum (\vec{r} \times \vec{v})_{out} \dot{m}_{out} - \sum (\vec{r} \times \vec{v})_{in} \dot{m}_{in}$$

3.16 The First Law of Thermodynamics and the Energy Equation

The first law of thermodynamics tells us:

Time rate of change of total system energy = Time rate of change by heat transfer + Time rate of change by work transfer

or

$$\left( \frac{dE}{dt} \right)_{sys} = \dot{Q} - \dot{W}$$

where $E$ is energy, $Q$ is heat, and $W$ is work. Recall that (‘) indicates time rate of change.

Sign conventions:

$\dot{Q}$ → is the transfer by radiation, conduction, convection of heat. Transfer into the control volume is positive.

$W > 0$ is work done by the system on the surroundings and $W < 0$ is work done on the system by the surroundings. Note that it is not uncommon to use the opposite sign convention for work, in which case the equation is written with the final terms as $+\dot{W}$ (you may have seen it this way in a thermodynamics course). In CEE 3310 we will always define $W > 0$ as work done by the system on the surroundings.

© 2015 Edwin A. Cowen
We can decompose the rate of work into components:

\[ \dot{W} = \dot{W}_{\text{shaft}} + \dot{W}_{\text{pressure}} + \dot{W}_{\text{viscous stress}} \]

where \( W_{\text{shaft}} \) is the work done by a machine such as a pump, turbine, piston, generator, etc.

\[ \dot{W}_{\text{shaft}} = T_{\text{shaft}} \Omega \]

where \( T_{\text{shaft}} \) is the torque and \( \Omega \) is the angular velocity.

Note that gravitational work will enter our energy budget through potential energy, which will show up linearly related to the height above a datum.

### 3.16.1 Stress Induced Work per Time (Power)

Work occurs by applying a force over a distance. Therefore pressure (normal stress) and shear (viscous stress) can produce work. If we look at the rate of work, or work per unit time, we have power and we see that we can think of this as applying a force on a system with a given velocity.

\[ \delta \dot{W} = \delta \vec{F} \cdot \vec{v} \]

Therefore for pressure we have

\[ \delta \dot{W}_{\text{pres}} = -P\vec{n} \delta A \cdot \vec{v} = -P\vec{v} \cdot \vec{n} \delta A \]

Therefore

\[ \dot{W}_{\text{pres}} = -\int_{CS} P\vec{v} \cdot \vec{n} \, dA \]

This term is usually moved to the right-hand-side flux term of the energy equation as it is a flux, which is how we will treat it.

For shear stress we have

\[ \delta \dot{W}_{\text{visc}} = \vec{\tau} \delta A \cdot \vec{v} = \vec{\tau} \cdot \vec{v} \delta A \]

Therefore

\[ \dot{W}_{\text{visc}} = \int_{CS} \vec{\tau} \cdot \vec{v} \, dA \]

© 2015 Edwin A. Cowen
But shear is internally self-canceling. On the control surface \( \vec{v} = 0 \) at solid boundaries (no-slip boundary condition), if the control surface is normal to the flow then \( \vec{r} \perp \vec{v} \) and hence \( \vec{r} \cdot \vec{v} = 0 \). Hence it is often reasonable to assume that the shear stress induced work is small.

What is a good environmental example of when this assumption breaks down?

Let’s consider the amount of work done!

### 3.17 The Energy Equation

With our definitions of work and energy we are now ready to apply the Reynolds Transport Theorem to produce the conservation of energy equation. Let \( B = E \) and \( \beta = e = E/m \), the energy per unit mass. We can decompose \( e \) into the following components

\[
e = \dot{u} + \frac{v^2}{2} + gz + \text{other}
\]

where \( \dot{u} \) is the internal energy per unit mass, \( v^2/2 \) is the kinetic energy per unit mass, and \( gz \) is the potential energy per unit mass. We will ignore other sources of internal energy but from the definition they are easily included.

We can write the Reynolds Transport Theorem flux term (including the pressure work term)

\[
\int_{CS} \left( \dot{u} + \frac{v^2}{2} + gz + \frac{P}{\rho} \right) \rho (\vec{v} \cdot \vec{n}) dA
\]

Therefore

\[
\left( \frac{dE}{dt} \right)_{sys} = \dot{Q} - \dot{W}_{shaft} = \frac{d}{dt} \int_{CV} \left( \dot{u} + \frac{v^2}{2} + gz \right) \rho d\Psi + \int_{CS} \left( \dot{u} + \frac{v^2}{2} + gz + \frac{P}{\rho} \right) \rho (\vec{v} \cdot \vec{n}) dA
\]

© 2015 Edwin A. Cowen
The 1-D form is
\[
\dot{Q} - \dot{W}_{\text{shaft}} = \frac{d}{dt} \int_{CV} \left( \dot{u} + \frac{v^2}{2} + gz \right) \rho \, dv + 
\sum_{\text{outflows}} \left( \dot{u} + \frac{v^2}{2} + gz + \frac{P}{\rho} \right) \dot{m} - \sum_{\text{inflows}} \left( \dot{u} + \frac{v^2}{2} + gz + \frac{P}{\rho} \right) \dot{m}
\]

We sometimes will choose to write \((\dot{u} + P/\rho) = \dot{h} = \text{enthalpy}\).

### 3.17.1 Forms of the Energy Equation

We often find it convenient to cast the energy equation in alternate forms.

**Velocity Squared form:**

If the flow is at steady state then conservation of mass gives us that \(\dot{m}_{\text{out}} = \dot{m}_{\text{in}} = \dot{m}\) and we can normalized all of our quantities by \(\dot{m}\) which leaves our homogeneous equation with terms having units of velocity squared

\[
\frac{\dot{Q}}{\dot{m}} - \frac{\dot{W}_{\text{shaft}}}{\dot{m}} = \dot{u}_{\text{out}} + \frac{v_{\text{out}}^2}{2} + g\dot{z}_{\text{out}} + \frac{P_{\text{out}}}{\rho} - \dot{u}_{\text{in}} - \frac{v_{\text{in}}^2}{2} - g\dot{z}_{\text{in}} - \frac{P_{\text{in}}}{\rho}
\]

Defining

\[
q = \frac{\dot{Q}}{\dot{m}} = \text{heat transfer unit mass} \quad \text{and} \quad w_s = \frac{\dot{W}_{\text{shaft}}}{\dot{m}} = \text{shaft work unit mass}
\]

we can write the above as

\[
\dot{u}_{\text{out}} + \frac{v_{\text{out}}^2}{2} + g\dot{z}_{\text{out}} + \frac{P_{\text{out}}}{\rho} = \dot{u}_{\text{in}} + \frac{v_{\text{in}}^2}{2} + g\dot{z}_{\text{in}} + \frac{P_{\text{in}}}{\rho} + q - w_s
\]

**Head form:**

Manometers have historically lead to a desire to think of energy in units of length, or *head*. We see that we can arrive at units of length by dividing our velocity squared form of the energy equation by gravity. Hence we have

\[
\frac{\dot{u}_{\text{out}}}{g} + \frac{v_{\text{out}}^2}{2g} + \frac{P_{\text{out}}}{\gamma} = \frac{\dot{u}_{\text{in}}}{g} + \frac{v_{\text{in}}^2}{2g} + \frac{P_{\text{in}}}{\gamma} + h_q - h_s
\]

where \(h_q\) and \(h_s\) are \(q/g\) and \(w_s/g\), respectively – the head forms of the heat transfer power and shaft power.

© 2015 Edwin A. Cowen
3.18 Incompressible 1-D Flow With No Shaft Work

Rearranging our 1-D head form and setting $w_s = 0$ we have

\[
\left( \frac{P}{\gamma} + \frac{v^2}{2g} + z \right)_{\text{out}} = \left( \frac{P}{\gamma} + \frac{v^2}{2g} + z \right)_{\text{in}} - \frac{\dot{u}_{\text{out}} - \dot{u}_{\text{in}} - q}{g}
\]

Now, defining \[\frac{\dot{u}_{\text{out}} - \dot{u}_{\text{in}} - q}{g} = h_{\text{loss}} = h_f\]

where we can think of $h_f$ as the friction losses and we see that $h_f > 0$ (note that in adiabatic flow, say a perfectly insulated pipe, friction will heat the flow, therefore $\dot{u}_{\text{out}} > \dot{u}_{\text{in}}$ and hence $h_f > 0$). Thus we write

\[
\left( \frac{P}{\gamma} + \frac{v^2}{2g} + z \right)_{\text{out}} = \left( \frac{P}{\gamma} + \frac{v^2}{2g} + z \right)_{\text{in}} - h_f
\]

3.18.1 Example – Gas Pipeline

Consider the following pipe flow:

If $Q = 75 \text{ m}^3/\text{s}$, the pipe radius is $r = 6 \text{ cm}$, the inlet pressure is maintained at 24 atm by a pump, the outlet vents to the atmosphere, the pipe rises 150 m from inlet to outlet.
and the pipe length is 10 km, what is $h_f$? What is the velocity head?

$h_f=198 \text{ m}$ Therefore the friction loss is greater than the $\Delta z$ and the pump must drive against both!

The velocity head is only 0.17 m!

Note that the length did not come into our solution. $h_f$ includes the total losses along the pipe due to friction effects and hence includes the effect of length implicitly. We will see more about this a bit later in the course.