7.3 Review

Minor Losses & Energy Equation example problems Non-Circular Pipes

- Laminar: \( f = \frac{64}{Re_D H} \pm 40\% \)
- Turbulent: \( f(Re_D, \epsilon/D_H) \Rightarrow \) Moody chart for \( f \pm 15\% \)

Bernoulli-Based Flow Metering

\[ Q = C_tA_t \sqrt{\frac{2\Delta P}{\rho(1-\beta^4)}} \]

where \( \beta = d/D \) and \( C_t = C_t(\beta, Re_D) \) is a coefficient determined by calibration.

7.4 Example – von Kármán’s Laminar Boundary Layer Problem

Von Kármán modeled the laminar flat-plate boundary layer after the fully developed laminar pipe flow solution - namely a parabolic profile. He modeled the flow as:

\[ u(x, y) = U_\infty \left( \frac{2y}{\delta} - \frac{y^2}{\delta^2} \right) \]

What is \( \tau_w \)? What is \( \delta \)?

\[ \Theta = \int_0^\delta \left( \frac{2y}{\delta} - \frac{y^2}{\delta^2} \right) \left( 1 - \frac{2y}{\delta} - \frac{y^2}{\delta^2} \right) dy = \frac{2}{15\delta} \]

Hence

\[ \tau_w = \mu \frac{\partial u}{\partial y} \bigg|_{y=0} = \frac{2}{\delta} U_\infty = \rho U_\infty^2 \frac{2}{15} \frac{d\delta}{dx} \]

Therefore

\[ \delta d\delta = 15 \frac{\mu}{U_\infty \rho} dx \]
Integrating and assuming that $\delta(0) = 0$ we find

$$\frac{\delta^2}{2} = \frac{15\nu x}{U_\infty}$$

Therefore we have found $\delta$. In nondimensional form we have

$$\frac{\delta}{x} = \sqrt{\frac{30\nu}{U_\infty x}} = \frac{5.5}{Re_x^{1/2}}$$

### 7.5 Skin Friction Coefficient

We can non-dimensionalize the wall stress by the dynamic pressure

$$c_f = \frac{\tau_w}{\frac{1}{2} \rho U^2}$$

where we call $c_f$ the *skin friction coefficient* (and it is clearly similar to the Euler number).

Hence, for von Kármán’s problem we have:

$$c_f = \frac{2\mu U}{\frac{1}{2} \rho U_\infty^2 \delta} = \frac{4\mu}{\rho U_\infty \sqrt{30\nu x}} = \sqrt{\frac{8\nu}{15 U_\infty x}} = \sqrt{\frac{8}{15 Re_x}}$$

Therefore

$$c_f \approx \frac{0.73}{Re_x^{1/2}}$$

As we will see in a minute, Kármán’s analysis is within 10% of the exact solution for laminar flows.