7.6 Review

Boundary layer Re

\[ \text{Re}_x = \frac{U_\infty x}{\nu} \gtrsim 10^6, \text{Turbulent} \]

Boundary layer height

\( \delta \) is defined such that \( u(\delta) = 0.99U_\infty \).

Momentum integral equation

\[ D(x) = \rho b U_\infty^2 \Theta(x) \]

where

\[ \Theta(x) = \int_0^{\delta(x)} \frac{u}{U_\infty} \left(1 - \frac{u}{U_\infty}\right) dy \]

7.7 Boundary Layer Equations

Prandtl is credited with simplifying the Navier-Stokes equations to a tractable form suitable for boundary layers. He argued:

- \( v \ll u \)
- \( \frac{\partial}{\partial x} \ll \frac{\partial}{\partial y} \)

Hence the \( y \)-momentum equation reduces to:

\[ 0 = -\frac{\partial P}{\partial y} \Rightarrow P = P(x) \]

The outer flow is assumed constant (note if it is not constant the Euler equations can be invoked to solve for the pressure field) hence \( P = P(x) = \) a constant. Hence the \( x \)-momentum equations become

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\rho} \frac{\partial \tau}{\partial y} \]
where

\[ \tau = \mu \frac{\partial u}{\partial y} \]

and hence

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \]

This is the boundary layer form of the momentum equations. Note that the 2-D continuity equation closes the system of equations. E.g.,

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

**The Laminar Velocity Profile - Blasius’ Solution**

Prandtl could not solve the above equation but Prandtl’s student Blasius used a clever similarity variable technique and solve the boundary layer equations for laminar flow (see other fluids texts for details - see, good students are sharper than their professors!).

A numerical solution to an ODE is required and a simple tabulated non-dimensional solution, known as the Blasius profile, is

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>( \frac{u}{u_\infty} )</th>
<th>( \eta )</th>
<th>( \frac{u}{u_\infty} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2.6000</td>
<td>0.7725</td>
</tr>
<tr>
<td>0.2000</td>
<td>0.0664</td>
<td>2.8000</td>
<td>0.8115</td>
</tr>
<tr>
<td>0.4000</td>
<td>0.1328</td>
<td>3.0000</td>
<td>0.8460</td>
</tr>
<tr>
<td>0.6000</td>
<td>0.1989</td>
<td>3.2000</td>
<td>0.8761</td>
</tr>
<tr>
<td>0.8000</td>
<td>0.2647</td>
<td>3.4000</td>
<td>0.9018</td>
</tr>
<tr>
<td>1.0000</td>
<td>0.3298</td>
<td>3.6000</td>
<td>0.9233</td>
</tr>
<tr>
<td>1.2000</td>
<td>0.3938</td>
<td>3.8000</td>
<td>0.9411</td>
</tr>
<tr>
<td>1.4000</td>
<td>0.4563</td>
<td>4.0000</td>
<td>0.9555</td>
</tr>
<tr>
<td>1.6000</td>
<td>0.5168</td>
<td>4.2000</td>
<td>0.9670</td>
</tr>
<tr>
<td>1.8000</td>
<td>0.5748</td>
<td>4.4000</td>
<td>0.9759</td>
</tr>
<tr>
<td>2.0000</td>
<td>0.6298</td>
<td>4.6000</td>
<td>0.9827</td>
</tr>
<tr>
<td>2.2000</td>
<td>0.6813</td>
<td>4.8000</td>
<td>0.9878</td>
</tr>
<tr>
<td>2.4000</td>
<td>0.7290</td>
<td>5.0000</td>
<td>0.9916</td>
</tr>
</tbody>
</table>
where $\eta = y \left( \frac{U}{\nu x} \right)^{0.5}$. From the Blasius profile one finds:

$$\frac{\delta}{x} = \frac{5.0}{Re_x^{1/2}}$$
and
$$c_f = \frac{0.664}{Re_x^{1/2}}$$

Hence Kármán’s analysis, based on the assumed parabolic form, is within 10%!

### 7.8 Turbulent Flat Plate Boundary Layer

If we return to dimensional analysis the only parameters that can be important in the boundary layer are:

$$\bar{u} = \phi(\rho, \nu, \tau_w, z)$$

Hence we have 5-3=2 dimensionless groups. We could select a Reynolds number based on the mean velocity and then nondimensionalize $\tau_w$ to form the second group or we recall from drag coefficients that we can define a new velocity - the friction velocity as

$$u^* = \sqrt{\frac{\tau_w}{\rho}}$$
and hence

$$\Pi_1 = \frac{\bar{u}}{u^*} = u^+$$
and
$$\Pi_2 = Re = \frac{u^* z}{\nu} = z^+$$

Thus we can write

$$u^+ = \psi(z^+)$$

Applying relevant local boundary conditions and taking the limits of viscous domination very near the wall and turbulence domination further from the wall, the functional form ($\psi$) of the solution to the above relationship can be found.

Very close to the wall where viscous effects are dominant the functional form is incredibly simple, $u^+ = z^+$.

A bit further from the wall where turbulence becomes dominant but the characteristic size of the turbulent eddies depends strongly on distance from the wall, the velocity profile form is:

$$u^+ = \frac{1}{\kappa} \ln z^+ + C$$
where $C$ is a constant generally taken to be 5.5 for high Reynolds number flows and $\kappa$ is von Kármán’s constant, generally taken to be 0.41. This form of the solution is known as the log-law or *law of the wall*.

We speak of the turbulent boundary layer as having various regions:

- $z^+ < 3.5$ viscous sublayer
- $3.5 < z^+ < 30$ buffer layer or transition layer
- $30 < z^+ \approx 200$ log-law
- $z/\delta > 0.2$ Outer layer and defect law region

The physical scale of $z^+$ is very small. As an example, in laboratory flumes a typical $u^*$ is about 5% of $\overline{u}$ (in natural rivers a range of 5-10% is typical). Hence for a flow of 20 cm/s $u^* \approx 1$ cm/s $\Rightarrow z^+ = 1$ occurs at $z = \nu/u^* = (0.01 \text{ cm}^2/\text{s})/(1 \text{ cm/s})= 0.1 \text{ mm}$! Thus the viscous sub-layer is confined to the bottom 0.5 mm of the flow. Often it is reasonable to approximate the turbulent boundary layer over a flat plate as dominated by the log-law region. An example of this is shown in the plot on the next page. You will see that for $y/\delta > 0.2$ the log-law model underestimates the actual velocity profile (here represented by the direct numerical simulation of the Navier Stokes equation carried out by Spalart at $Re_\theta = 1410$ – this is the defect region and to do a better job we would need to model what is known as the velocity defect ($U_\infty - \overline{u}$).


An even simpler model of the turbulent boundary layer mean velocity profile over a flat plate is a power law formulation:

$$\frac{\overline{u}}{U_\infty} = \left(\frac{y}{\delta}\right)^n$$

where $n$ is typically taken to be $1/7$. In reality we expect $n$ to depend on the Reynolds number and it does - it increases with increasing Re. Hence for low Re turbulence $n = 1/6$ is often a better fit as shown in the figure below.
7.9 The Flat Plate Boundary Layer Velocity Profile

Let’s conclude with a look at the flat plate boundary layer velocity profiles and the models. We have Blasius’s exact solution for Laminar flow. For turbulent flow to obtain an exact solution we must resort to a computational solution of the full Navier Stokes solutions. This calculation was performed by Philippe Spalart (1986). Direct simulation of a turbulent boundary layer up to $Re_\theta=1410$. NASA Technical Memorandum 89407.

Note that Spalart’s calculation was for relatively low Re and hence the power law with best fit is about $n = 1/6$ and the constant $C$ in the log-law is 5.0. It is clear that the Blasius solution is very close to a simple parabola (e.g., Kármán’s solution). Also, for the inner region of the turbulent boundary layer the log-law works exceptionally well. Hence for lower elevations in the atmosphere (say less than a few hundred meters from the ground) the log-law works very well.
7.10 BL Example