8.10 Review

Open Channel Flow $\Rightarrow$ Gravity – friction balance.

Uniform Flow $\Rightarrow \frac{\partial y}{\partial x} = 0 \quad \Delta z = S_0 L = h_f$

Rapidly Varied Flow $\Rightarrow \frac{\partial y}{\partial x} \sim 1$

Gradually Varied Flow $\Rightarrow \frac{\partial y}{\partial x} \ll 1$

In general we take an energy equation approach.

In uniform flow $\Delta z = S_0 L = h_f$

We can apply Darcy-Weisbach with $D_H$ defined based on wetted perimeter.

Summary of what we will cover today from last lecture’s notes:

\[ V = C (R_h S_0)^{1/2} \]

Chézy formula where $R_h = D_h/4$

where $C = \left( \frac{8g}{f} \right)^{1/2}$ is the Chézy coefficient

which is tabulated or can be found from the Moody diagram using $f(Re_{D_h}, \epsilon/D_h)$ where we could find $h_f$ from the Darcy-Weisbach friction factor and $f(Re_{D_h}, \epsilon/D_h)$

Manning found $C = \left( \frac{8g}{f} \right)^{1/2} \approx \alpha \frac{R_h^{1/6}}{n}$

and hence Manning’s equation:

\[ V = \frac{\alpha}{n} R_h^{2/3} S_0^{1/2} \] and for $b \gg d$, $R_h \approx d \Rightarrow V \approx \frac{\alpha}{n} d^{2/3} S_0^{1/2}$ and $Q = V A \approx \frac{\alpha}{n} b d^{5/3} S_0^{1/2}$

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8.11 Energy Equation - Revisited

Our last work with the energy equation left us with:

\[
\frac{V_1^2}{2g} + z_1 = \frac{V_2^2}{2g} + z_2 + h_L
\]

But we can write \( h_L = S_f L \). Defining \( \xi = z - y \) we have \( \xi_1 = \xi_2 + S_0 L \) and

\[
\frac{V_1^2}{2g} + y_1 + S_0 L = \frac{V_2^2}{2g} + y_2 + S_f L
\]

hence

\[
\frac{V_1^2}{2g} + y_1 = \frac{V_2^2}{2g} + y_2 + (S_f - S_0)L
\]

8.11.1 Specific Energy

Define

\[
E = y + \frac{V^2}{2g}
\]

which is known as the specific energy.

The specific energy is seen to be the height of the EGL above the bed. Let \( q = Q/b = V y \)
where \( b \) is the width of the channel. We can express the specific energy as

\[
E = y + \frac{q^2}{2gy^2}
\]

We see that this curve is a cubic in \( y \) – let’s look at the shape of this plot:

Let’s look at the minimum in \( E \). We can find its location (which we will denote \( y_c \), the critical depth) by solving

\[
\frac{\partial E}{\partial y} = 0 = 1 - \frac{2q^2}{2gy_c^3} \quad \Rightarrow \quad y_c = \left( \frac{q^2}{g} \right)^{1/3} = \left( \frac{Q^2}{b^2g} \right)^{1/3}
\]
Now we can define the minimum in the specific energy curve \( (E_{\text{min}}) \)

\[
E_{\text{min}} = E(y_c) = \left( \frac{q^2}{g} \right)^{1/3} + \frac{q^2}{2g} \frac{q^{4/3}}{g^{2/3}} = \frac{q^{2/3}}{g^{1/3}} + \frac{q^{2/3}}{2g^{1/3}} = \frac{3q^{2/3}}{2g^{1/3}} = \frac{3}{2} y_c
\]

Defining \( q = V y = V y_c \) we can substitute into our definition of \( y_c \)

\[
y_c^3 = \frac{q^2}{g} = \frac{V_c^2 y_c^2}{g} \quad \Rightarrow \quad V_c^2 = g y_c \quad \Rightarrow \quad V_c = \sqrt{g y_c}
\]

Recall from Lab #3 and dimensional analysis that

\[
Fr = \frac{V}{\sqrt{g y}} \quad \Rightarrow \quad Fr_c = \frac{V_c}{\sqrt{g y_c}} = \frac{\sqrt{g y_c}}{\sqrt{g y_c}} = 1
\]

Hence at the inflection point in the specific energy curve we have

\[
Fr = 1, \quad y = y_c, \quad E = E_{\text{min}} = \frac{3}{2} y_c
\]

thus the inflection point separates supercritical flow from sub-critical flow (recall Lab #3 for the definitions of super- and sub-critical flow). The upper portion of the specific energy curve is known as the subcritical branch (deeper, slower flows at a given energy) while the lower branch is known as the supercritical branch) shallower, faster flows at a given energy).
8.12 Example - Flow Over a Sill (Bump)

Consider the following constant width channel:

If we take the bed to be horizontal \((S_0 = 0)\) away from the sill and the flow to be frictionless \((S_f = 0)\) then the Bernoulli form of the specific energy equation gives us (where \(q\) is a constant of the flow since the flow width is constant):

\[
E_1 = E_2 + \Delta h = y_2 + \frac{q^2}{2gy_2^3} + \Delta h
\]

Rearranging to be a polynomial in \(y_2^2\) we find:

\[
y_2^3 + (\Delta h - E_1)y_2^2 + \frac{q^2}{2g} = 0
\]

The coefficient terms to the polynomial are all known \((E_1\) is the left-hand side boundary condition and hence \(y_1\) and \(q\) and thus \(E_1\) are all known for a well specified problem). Clearly this equation has three roots. It turns out that for \(\Delta h\) not too large it has 3 real roots, one of which is negative (and hence is not physically possibly as a negative water depth does not make sense), leaving two real positive roots that are viable flow depths. Let’s consider the range of possibilities for this flow.

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a) Initial Flow Subcritical Upstream and Downstream (e.g., Fr_1 < 1 and Fr_3 < 1)

b) Initial Flow Supercritical Upstream and Downstream (e.g., Fr_1 > 1 and Fr_3 > 1)

c) Initial Flow Subcritical Upstream and Supercritical Downstream (e.g., Fr_1 < 1 and Fr_3 > 1)

The specific energy plot for each flow looks like:

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8.13 Example - Flow though a Contraction

Consider the following channel of constant bottom elevation with vertical banks:

As in our sill example we take the bed to be horizontal \((S_0 = 0)\) and the flow to be frictionless \((S_f = 0)\). However, now \(q\) is no longer constant throughout the flow as \(b = b(x) \Rightarrow q = Q/b = q(x)\). We consider the same three cases:

a) \(Fr_1 < 1\) and \(Fr_3 < 1\)

b) \(Fr_1 > 1\) and \(Fr_3 > 1\)
c) \( Fr_1 < 1 \) and \( Fr_3 > 1 \)

The specific energy plot for each flow looks like:

![Specific Energy Plot](image-url)