3.31 Review

• A special case of the Energy Equation – The Bernoulli Equation: Applicable if you are on a streamline, at steady state, the flow can be well approximated as frictionless and no work or heating is done along the streamline

\[ \frac{P}{\gamma} + \frac{v^2}{2g} + z = h_0 \]

Can think of it as the inviscid (frictionless) conservation of energy equation (i.e., \( h_f = 0 = h_s \)).

• Stagnation pressure – the sum of the static pressure and the dynamic pressure.

\[ P_s = P + \rho \frac{v^2}{2} \]

• Pitot-static tube \(- V = \sqrt{\frac{2 \Delta P}{\rho}} = \sqrt{2 \frac{\gamma \Delta H}{\rho}} = \sqrt{2 g \Delta H} \), a particular case of the Bernoulli equation, where \( \Delta P \) or \( \Delta H \) is the measured differential pressure or head, respectively, between the stagnation and static pressures.

• Torricelli’s Law \(- V = \sqrt{2gh} \), a particular case of the Bernoulli equation.

• Energy grade line (EGL) – the total energy in head form as a function of position along the system.

• Hydraulic grade line (HGL) – the non-dynamic energy in head form as a function of position along the system.

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3.32 Irrotational Flow and Bernoulli

Consider

\[
\frac{P}{\gamma} + \frac{v^2}{2g} + z = h_0
\]
\[
0 + \frac{v^2}{2g} + 0 = h_0 = \frac{v^2}{2g}
\]
\[
\frac{h_2}{\gamma} + \frac{v^2}{2g} - h_2 = h_0 \sqrt{\gamma}
\]
\[
\frac{h_3}{\gamma} + \frac{v^2}{2g} - h_3 = h_0 \sqrt{\gamma}
\]
\[
\frac{h}{\gamma} + 0 - h = 0 \neq h_0
\]

Notice that in the unsheared regions (uniform flow) \( h_0 \) = a constant across streamlines while where shear exists (e.g., shear is non-zero), \( h_0 \) varies across the streamlines. More strictly speaking we actually want to know if the flow is \textit{rotational}. Our test is if we stick a small neutrally buoyant + shaped probe in the flow and see if it will rotate. In uniform flow it will not, in a linear shear, like the shear profile shown here, it will. Hence we say that \( h_0 \) is constant in irrotational (non rotational) flows. This allows us to connect Bernoulli points that are not on the same streamline in flows that are irrotational, further expanding the power of the Bernoulli equation but also the opportunities to misuse it!
3.33 Cavitation and the Bernoulli Equation

Consider the following flow geometry:

If we assume over such a short section friction is negligible then, given the constant $z$, the Bernoulli equation reduces to

$$P_0 = P + \rho \frac{v^2}{2}$$

Therefore high pressure occurs when the velocity is low and as the velocity increases the pressure drops. Plotting the variation of the dynamic and static pressure:

we find that the pressure might fall below the vapor pressure of the fluid and hence cavitate. Thus in general we should be concerned about cavitation any time we are dealing with relatively high velocities.
Chapter 4

Differential Analysis

4.1 Acceleration Revisited - The Eulerian Perspective

We define the acceleration of a particle or fluid parcel as

$$a_x = \frac{du}{dt}, \quad a_y = \frac{dv}{dt}, \quad a_z = \frac{dw}{dt}$$

where

$$u = \frac{dx}{dt}, \quad v = \frac{dy}{dt}, \quad w = \frac{dz}{dt}$$

clearly this is a Lagrangian description – we are describing the acceleration of a fluid parcel or particle which is moving with the flow. It is also apparent that the Eulerian and Lagrangian descriptions of velocity are identical, namely the time rate of change of position with respect to some point. Now what is the acceleration in an Eulerian reference frame?

We write the velocity as

$$u = u(x, y, z, t), \quad v = v(x, y, z, t), \quad w = w(x, y, z, t) \quad \text{and} \quad \vec{v} = u\vec{i} + v\vec{j} + w\vec{k}$$
and by the chain rule we have

\[
\frac{da_x}{dt} = \frac{du}{dt} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}
\]

\[
= \frac{\partial u}{\partial t} + \vec{u} \cdot \nabla u = \frac{Du}{Dt}
\]

where we recall that \( \frac{D}{Dt} \) is the substantial derivative. Similarly

\[
\frac{da_y}{dt} = \frac{dv}{dt} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z}
\]

\[
= \frac{\partial v}{\partial t} + \vec{u} \cdot \nabla v = \frac{Dv}{Dt}
\]

\[
\frac{da_z}{dt} = \frac{dw}{dt} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}
\]

\[
= \frac{\partial w}{\partial t} + \vec{u} \cdot \nabla w = \frac{Dw}{Dt}
\]

Therefore,

\[
\vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k} = \frac{Du}{Dt} \vec{i} + \frac{Dv}{Dt} \vec{j} + \frac{Dw}{Dt} \vec{k} = \frac{D\vec{u}}{Dt}
\]

4.1.1 The Substantial Derivative – an Example

If a hiker starts at 10,000 feet on a 20,000 foot hike at 8 am and is hiking to 14,000 feet and a reasonable model of the atmosphere is the standard atmosphere and a reasonable model of the diurnal effects of sun on temperature is \( \Theta = \Theta_{8am} + \alpha t \) where \( \alpha = 3^\circ C/hr. \)

What velocity over ground must be sustained in order to experience a constant temperature?
0.8 mph.

4.2 Differential Analysis - Conservation of Mass

Consider the infinitesimal fixed control volume

Therefore

\[ \int_{CV} \frac{\partial \rho}{\partial t} \, dV + \sum_i (\rho_i A_i v_i)_{out} - \sum_i (\rho_i A_i v_i)_{in} = 0 \]

Now, the control volume is infinitesimal so we can take \( \frac{\partial \rho}{\partial t} \) to be spatially constant. Therefore

\[ \int_{CV} \frac{\partial \rho}{\partial t} \, dV = \frac{\partial \rho}{\partial t} \, dx \, dy \, dz \]

4.2.1 Flux Terms

\[ \left[ \rho u + \frac{\partial (\rho u)}{\partial x} \right] \, dx \, dy \, dz \]

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If we extend our analysis to all three faces we find a total of three inflows and three outflows.

<table>
<thead>
<tr>
<th>Face</th>
<th>( \dot{m}_{in} )</th>
<th>( \dot{m}_{out} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( \rho u , dy , dz \left( \rho u + \frac{\partial (\rho u)}{\partial x} , dx \right) , dy , dz )</td>
<td></td>
</tr>
<tr>
<td>( y )</td>
<td>( \rho v , dx , dz \left( \rho v + \frac{\partial (\rho v)}{\partial y} , dy \right) , dx , dz )</td>
<td></td>
</tr>
<tr>
<td>( z )</td>
<td>( \rho w , dx , dy \left( \rho w + \frac{\partial (\rho w)}{\partial z} , dz \right) , dx , dy )</td>
<td></td>
</tr>
</tbody>
</table>

But \( \dot{m}_{in} > 0, \dot{m}_{out} < 0 \), therefore

\[
\frac{\partial \rho}{\partial t} \, dx \, dy \, dz + \frac{\partial (\rho u)}{\partial x} \, dx \, dy \, dz + \frac{\partial (\rho v)}{\partial y} \, dx \, dy \, dz + \frac{\partial (\rho w)}{\partial z} \, dx \, dy \, dz = 0
\]

Therefore

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0
\]

This is the conservation of mass equation written in differential form, also known as the continuity equation. We can also write it

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0
\]

### 4.2.2 Steady Flow

\[
\frac{\partial}{\partial t} = 0 \quad \Rightarrow \quad \nabla \cdot (\rho \vec{v}) = 0
\]

### 4.2.3 Steady, Incompressible Flow

\[
\frac{\partial \rho}{\partial t} = 0, \quad \rho = \text{constant}
\]

Therefore

\[
\rho (\nabla \cdot \vec{v}) = 0 \quad \text{and hence} \quad \nabla \cdot \vec{v} = 0 \quad \text{or} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]
4.2.4 Example – water waves

Consider what is known as a linear deep water wave (a wave traveling in a fluid whose depth is sufficiently large that the wave does not know that there is any bottom). The equation for the $x-$component of the velocity of a wave traveling in the $x$ direction is:

$$u(x, z, t) = a\sigma \cos (kx - \sigma t)e^{kz}$$

where $k = 2\pi/\lambda$, $\lambda$ is the wavelength of the water wave, $\sigma = 2\pi/\lambda$, and $T$ is the wave period. If the wave is 2-D (e.g., $v = 0$ and $\partial/\partial y = 0$), and that water is incompressible, what is the vertical velocity component?

$$w(x, z, t) = a\sigma \sin (kx - \sigma t)e^{kz}$$

If you are intrigued by water waves consider taking CEE 4350 in spring 2018 (it may be offered - no promises)!