

Chapter 2

Hydrostatics

2.1 Review

- Contact angle: defined as the angle between the tangent to the free surface directed from the solid-liquid-gas interface to the solid boundary through the liquid.
- Surface tension forces at solid/liquid/gas intersections (the contact line) are found by taking the component of the tension force (contact line length times surface tension) in the direction of interest.
- *Eulerian* – description from the perspective of fixed points within a reference frame.
- *Lagrangian* – description from the perspective of a parcel moving within the flow.
- *Streamline* – Eulerian, tangent line to instantaneous velocity field.
- *Pathline* – Lagrangian, path of a fluid parcel.
- *Streakline* – Locus of parcels that have passed through a given point.

2.2 Hydrostatics and Pressure

In many fluid problems the velocity is zero or the velocity is constant $\Rightarrow \tau = 0$.

If the tangential stresses are zero this leaves only the normal stresses, which in the absence of acceleration arise only from the *pressure*. Then our question is one of *statics*, meaning we have no motion (or relative motion), and how does the pressure vary with direction?

The answer is *the pressure at a point is independent of direction!* - see section 2.1 in your textbook for the derivation. Hence we speak of pressure as a *scalar* quantity as it is a quantity with no dependence on direction (as opposed to velocity, which is a vector).

This result is known as *Pascal's law* and says that if $\tau = 0$ the pressure at a point is the same in all directions.

2.3 Equation of Motion in absence of Shear Stresses

Let's investigate the force balance on a small volume of fluid to determine the equation of motion in the absence of shear stresses

Surface Forces

$$\frac{d\vec{F}_s}{d\mathcal{V}} = -\nabla P = \frac{\text{Force}}{\text{Volume}}$$

Body Forces

What body forces might act on the fluid element? Gravity, electromagnetic, ...

$$\Delta\vec{F}_g = \rho\vec{g}\Delta x\Delta y\Delta z$$

Taking the limit as Δx , Δy , $\Delta z \rightarrow 0$ we have

$$\frac{d\vec{F}_g}{d\mathcal{V}} = -\gamma\vec{k}$$

Now, putting it all together we have

$$\begin{aligned}\vec{F} &= m\vec{a} \\ \frac{d\vec{F}_s}{dV} + \frac{d\vec{F}_g}{dV} &= \rho\vec{a} \\ -\nabla P - \gamma\vec{k} &= \rho\vec{a}\end{aligned}$$

2.4 Hydrostatic Pressure

We found the equation of motion for a fluid with $\tau = 0$:

$$-\nabla P - \gamma\vec{k} = \rho\vec{a}$$

Now, if the fluid is at rest (or at least moving at a constant velocity):

$$\vec{a} = 0 \quad \Rightarrow \quad \nabla P = -\gamma\vec{k}$$

Hence we can write:

$$\frac{\partial P}{\partial x} = 0, \quad \frac{\partial P}{\partial y} = 0, \quad \frac{\partial P}{\partial z} = -\gamma$$

What does this tell us? Well, we see that $P = P(z)$ only, hence the pressure at a given elevation (z position) is constant. In the vertical we have

$$\frac{dP}{dz} = -\gamma \quad \text{where we have replaced } \partial \text{ with } d \text{ now.}$$

Hence as $z \uparrow$ $P \downarrow$

2.4.1 Incompressible Fluids

$$dP = -\gamma dz \quad \Rightarrow \quad \int_{P_1}^{P_2} dP = - \int_{z_1}^{z_2} \gamma dz \quad \Rightarrow \quad P_2 - P_1 = - \int_{z_1}^{z_2} \gamma dz$$

Now let's consider particular solutions:

Constant γ

It is usually reasonable to assume that all liquids essentially have a constant γ , certainly true for all fluids at constant temperature and pressure. Under this condition we have

$$P_2 - P_1 = -\gamma \int_{z_1}^{z_2} dz = -\gamma(z_2 - z_1) = -\gamma h$$

Therefore

$$P_1 = \gamma h + P_2 \quad \text{or} \quad h = \frac{P_1 - P_2}{\gamma}$$

where we refer to h as the *pressure head* as it is a pressure measured in units of length.

Liquids with a Free Surface

Examples include lakes, rivers, oceans, reservoirs, your pint glass of water, ...

The surface is the datum (at this point the pressure is the pressure of the atmosphere or surrounding gas). Therefore we have $z = 0$, $P = P_{atm}$.

$$P_1 = \gamma h + P_{atm} \quad \text{but} \quad h = z_2 - z_1 = -z_1 \quad \Rightarrow \quad P = P_{atm} - \gamma z$$

where $z < 0$ in liquid (water) and $z > 0$ in gas (atmosphere).

2.4.2 Compressible Gases

$$\frac{dP}{dz} = -\gamma = -\rho g = -\frac{P}{R\Theta}g$$

How do we solve this? \Rightarrow separation of variables!

$$\int_{P_1}^{P_2} \frac{dp}{P} = - \int_{z_1}^{z_2} \frac{g dz}{R\Theta}$$

we assume g is a constant but that Θ may vary (e.g., the atmosphere). Then we can write the above

$$\ln \frac{P_2}{P_1} = -\frac{g}{R} \int_{z_1}^{z_2} \frac{dz}{\Theta}$$

If we assume that the temperature is constant within the elevation range of interest, an assumption known as isothermal conditions (i.e., $\Theta = \Theta_0 = \text{a constant}$), we can write

$$\ln \frac{P_2}{P_1} = -\frac{g}{R\Theta_0}(z_2 - z_1) \quad \text{or} \quad P_2 = P_1 \exp\left(-\frac{g(z_2 - z_1)}{R\theta_0}\right)$$

2.4.3 The Standard Atmosphere

In the troposphere (first 11.0 km of atmosphere) the temperature drops linearly with elevation. The rate of temperature drop is known as the lapse rate and is given by $\beta = 0.00650 \text{ K/m}$ (6.5 °C/km or 3.6 °F/1000'). Hence in the troposphere we write

$$\Theta = \Theta_a - \beta z$$

where Θ_a is the temperature at sea level which for the standard atmosphere is taken to be 15°C. We can use the expression we found for compressible gasses (recall we allowed for temperature variation!) and integrate to find

$$P = P_a \left(1 - \frac{\beta z}{\Theta_a}\right)^{\frac{g}{R\beta}}$$

This is the expression for the pressure in the troposphere.

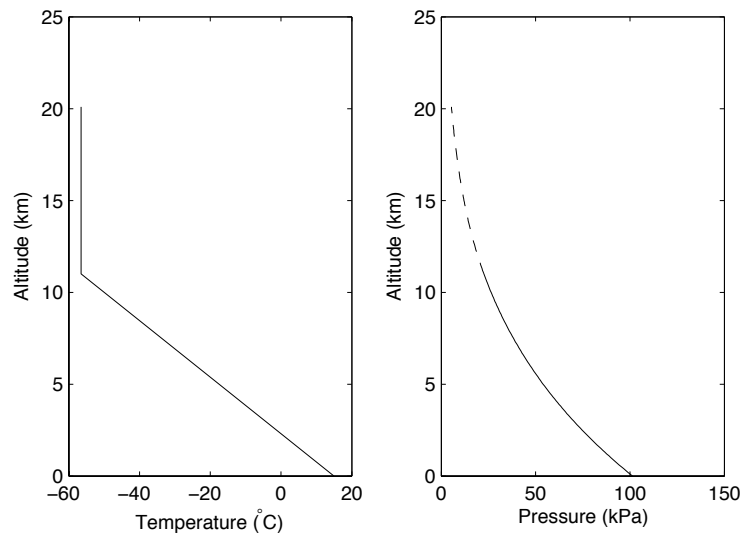


Figure 2.1: U.S. standard atmosphere to 20.1 km

2.5 Measurement of Pressure

Absolute pressure is the pressure relative to a perfect vacuum, hence it is always a positive (or zero) value.

Gage pressure is the pressure relative to the local atmospheric value

$$P > P_a \Rightarrow P_{\text{gage}} = P - P_a > 0$$

$$P < P_a \Rightarrow P_{\text{gage}} = P - P_a < 0 \quad P_{\text{vacuum}} = P_{\text{suction}} = P_a - P > 0$$

Units of pressure are force/area hence pounds per square inch (psi), pounds per square foot (psf), newtons per square meter (N/m^2 =pascal (Pa)).

We will take all pressures, unless otherwise noted, to be gage.

Example – Liquids with a Free Surface

2.5.1 Barometer

$$P_{\text{atm}} - P_{\text{vapor}} = \gamma h \quad \Rightarrow \quad P_{\text{atm}} = \gamma h + P_{\text{vapor}}$$

For mercury $P_{\text{vapor}} \sim 0 \Rightarrow P_{\text{atm}} = \gamma h$.

Typical values of pressure (from the U.S. standard atmosphere at sea level) are: 14.7 psia, 101 kPa, or, 760. mm Hg, 29.9 in Hg, 33.9 ft water (recall $\gamma_{\text{water}} = 62.4 \text{ lbs/ft}^3$) and $\text{S.G.}_{\text{mercury}} = 13.6$).

2.6 Manometry

A *manometer* is a vertical or inclined tube used to measure pressure. There are three fundamental types of manometers: Piezometer tube, U-tube, and inclined tube.

Manometer analysis is straight forward hydrostatics - the key is to keep track of the signs of the pressure terms! Consider the following U-tube manometer:

$$P_2 - P_1 = -\gamma(z_2 - z_1)$$

Now we track the pressure from A to 1 and from 1 to 2. A simple rule, based on our understanding that as we move down in a static fluid the pressure increases, is:

$$P_{\text{down}} = P_{\text{up}} + \gamma|\Delta z|$$

This obviates the need to keep track of signs of directions in the vertical. Hence for our U-tube problem we have

$$P_A + \gamma_1|z_A - z_1| - \gamma_2|z_2 - z_1| = P_{\text{atm}}$$

and we solve for P_A

Example - Inclined Manometer