

2.7 Review

- Pressure is a scalar - independent of direction.
- Hydrostatics: $\nabla P = -\gamma \vec{k}$. Horizontal motion in the same fluid - no change in pressure, vertical motion: $z \uparrow P \downarrow$, $z \downarrow P \uparrow$.
- $\frac{\partial P}{\partial z} = -\gamma$ pressure is constant at $z = \text{constant}$

2.8 Hydrostatic Force on a Plane Surface

Consider the force on a horizontal plate due to a fluid of depth H

We know that over the entire plate (with surface area A) the pressure is constant (since we do not change our z position) hence we can write $F_R = PA$ where F_R is known as the *resultant* force, as the net of the pressure acting on the plane surface is equivalent to the resultant force applied at a particular location (which can be shown to be the centroid of the area). Recall that for the case shown (a rectangular fish tank) $P = \gamma h \Rightarrow F_R = \gamma h A = \gamma V = \text{the weight of the fluid!}$ Thus the pressure force on the bottom of the tank is just the weight of the fluid which makes sense!

What about a surface inclined to the horizontal? Consider

We start with the differential form of our statement above about the force on an area, namely:

$$dF_R = P dA \quad \text{which we integrate to find} \quad F_R = \int_A P dA$$

In a hydrostatic fluid we expect no variation of P with position in the x direction but we do expect P to vary as the depth changes (e.g., as we move in z).

If we consider a 2-D Cartesian coordinate system with its origin at the centroid of our surface (\mathcal{S}), the y coordinate oriented vertically downward (this is the opposite as is in your text book which defines y vertically along the surface positive upwards) along the surface (i.e., in the plane of the surface maximally aligned with gravity – note we are now using y as the vertical coordinate along the surface and it is *in the direction of gravity*, as opposed to z which is in the *opposite* direction of gravity, and, in particular, is still truly vertical) and the x direction normal to the y direction (i.e., horizontal) on the surface and set by the right-hand-rule, we can write this

$$F_R = \int \int_{\mathcal{S}} P(x, y) dx dy \tag{2.1}$$

In stating the above all we have assumed is:

1. The fluid is hydrostatic ($u = \text{constant}$).
2. The surface \mathcal{S} is planar.

Thus for flat surfaces in hydrostatic fluids the above is always true.

If we now allow ourselves one more assumption, namely:

3. The density over the surface \mathcal{S} is constant (this may be relaxed later, as discussed in the text in detail).

Then we may write

$$F_R = P_C A \quad (2.2)$$

where

$$P_C = \frac{1}{A} \int \int_{\mathcal{S}} P(x, y) \, dx dy \quad (2.3)$$

hence P_C is just the average pressure acting over the surface \mathcal{S} which we see is mathematically the same as the pressure acting at the centroid of the surface \mathcal{S} and A is the area of the surface \mathcal{S} . Thus our only restriction in applying equation 2.2 is that the fluid must have a constant density wherever it contacts \mathcal{S} . Setting equations 2.1 and 2.2 equal yields equation 2.3 confirming that the pressure acting at the centroid (P_C) is equivalent to the average pressure acting over the surface \mathcal{S} .

Therefore the magnitude of the force depends on

- P_C – the pressure acting at the centroid of the surface \mathcal{S} .
- A – the area of the surface \mathcal{S} .

Now, all we need to do is calculate the pressure at the centroid of our surface, P_C , which is easy to do! For example, if we have a surface \mathcal{S} inclined at an angle θ to the horizontal overlaid by a fluid of constant specific weight γ with a free surface above it at a pressure P_A then we have

$$F_R = P_A A + \gamma h_C A = (P_A + \gamma h_C) A = P_C A$$

where h_C is the depth of fluid over the centroid of the surface \mathcal{S} , which is found to be

$$h_C = \xi_C \sin \theta$$

where ξ_C is the distance parallel to the surface \mathcal{S} measured from the centroid to the free surface, e.g.,

2.8.1 Where is the force located on the surface?

Why doesn't the resultant force just act at the centroid of the area?

O.K., let's rigorously determine where the force acts.

The location that the force acts on the surface is known as the *center of pressure* and is denoted y_R as it is the location that the resultant force, F_R , acts on the surface.

Based on the definition of pressure, the force is clearly directed perpendicular to the surface \mathcal{S} , and from the fluid toward the surface.

Now, recalling from mechanics that the moment of the resultant force about any point is equal to the sum of the moments of the component forces about the same point, we

can write

$$y_R F_R = \int_A y P dA = \int_A y (P_C + \gamma y \sin \theta) dA = P_C \int_A y dA + \gamma \sin \theta \int y^2 dA$$

But the first term on the right-hand-side is just the first moment of A which occurs at the centroid and hence in our coordinate system this is just $= 0$! Hence

$$y_R F_R = \gamma \sin \theta \int y^2 dA$$

recalling that $F_R = P_C A$ and solving for y_R

$$y_R = \frac{\gamma \sin \theta}{P_C A} \int y^2 dA$$

The integral in the above expression is the second moment of the area with respect to the x axis through the centroid (horizontal axis, or the line perpendicular to gravity through the centroid of the surface \mathcal{S}) and is denoted I_{xc} , hence we arrive at our final expression for y_R

$$y_R = \frac{I_{xc} \gamma \sin \theta}{P_C A} \quad (2.4)$$

where

$$I_{xc} = \int y^2 dA \quad (2.5)$$

We make two important findings based on this last equation:

1. The above is positive definite hence we find that y_R is always positive which in our coordinate system means that it always acts *below* the centroid, which is located at $y = 0$ by definition.
2. As P_C gets larger for a given surface \mathcal{S} with area A , the value of y_R gets smaller indicating that the deeper we go the closer the center of pressure is to the centroid of the surface \mathcal{S} .

2.8.2 x -position of Center of Pressure

Now, we follow the same approach as previously used (taking the moments) to get x_R , the x position of the center of pressure:

$$x_R F_R = \int_A x P dA = \int_A x (P_C + \gamma y \sin \theta) dA = P_C \int_A x dA + \gamma \sin \theta \int xy dA$$

But the first term on the right-hand-side is again just the first moment of A (with respect to x this time) which occurs at the centroid (e.g., $x_C = 0$ in our coordinate system) and hence is zero! Therefore we have

$$x_R F_R = \gamma \sin \theta \int xy dA$$

recalling that $F_R = P_C A$ and solving for x_R

$$x_R = \frac{\gamma \sin \theta}{P_C A} \int xy dA \quad (2.6)$$

The integral in the above expression is known as the product of inertia (a mixed second moment) of the area and is denoted I_{xyc} . It's values, along with I_{xc} can be found in standard tables (Page 80 of your text and a link on the course web site, for example). It is essentially a measure of the asymmetry of the surface – surfaces that are symmetric about either axis, which pass through the centroid by definition, will have $I_{xyc} = 0$. Our final expression for x_R is

$$x_R = \frac{I_{xyc} \gamma \sin \theta}{P_C A}$$

Since shapes of interest often obey symmetry about an axis through the centroid we will often find that $x_R = 0$.

Summarizing our findings for the case when the density over the surface \mathcal{S} is constant:

$$F_R = P_C A \quad (2.7)$$

$$x_R = \frac{I_{xyc} \gamma \sin \theta}{P_C A} \quad (2.8)$$

$$y_R = \frac{I_{xc} \gamma \sin \theta}{P_C A} \quad (2.9)$$

Note these are more general than those presented in the text (Eq. 2.29) which makes the assumption that the ambient pressure is acting on both sides of the plate which may not be always true! We will have examples where it is not so caveat emptor!