Chapter 2

Canonical Turbulent Flows

Before we consider two canonical turbulent flows we need a general description of turbulence.

2.1 A Brief Introduction to Turbulence

One way of looking at turbulent flow is to consider it to be subject to two types of random movement: one on the scale of the thermal motion of molecules and one on the scale (macroscopic) of fluid velocity fluctuations (eddies). The main difference between these two motion types is:

1. Scale—the turbulent “fluctuation” scale is typical $\gg$ molecular mean free path.

2. Continuity—in a turbulent fluctuation some “chunk” of fluid must be pushed out of the way for a turbulent fluctuation to occur (and similarly, fluid must replace the void left by the mass of fluid carried in the turbulent fluctuation).

How do we know a fluid is turbulent? Symptoms of turbulence:
• Diffusive (more so than due to molecular diffusion, much much more so . . .).

• Unsteady ⇒ “Random” (there is coherence in turbulence so it is not truly random).

• Wide range of scales of motion

Consider the following:

2.1.1 The Energy Spectrum

“Big whirls have little whirls which feed on their velocity; and little whirls have lesser whirls, and so on to viscosity.”

L.F. Richardson, c. 1920

The Cascade Picture

The energy of an eddy ∼ u² where u is the typical (read mean) velocity at a location.
Let’s assume that an eddy passes all of its energy to smaller scales in the time it takes the eddy to rotate approximately one revolution, known as the eddy turnover time. Then the rate of loss of energy to smaller scales $\Rightarrow \frac{u^2}{T_e}$

Where $T_e$ is the eddy turnover time $\sim \frac{L}{u}$ ($L$ is the diameter, or length scale, of the typical eddy).

Therefore the rate of loss of energy to the smallest scales $\sim \frac{u^3}{L} = \varepsilon = \text{dissipation rate of turbulent kinetic energy.}$

If the flow is in equilibrium (meaning the rate of the production of turbulence is equal to the rate of the dissipation of turbulence locally), this energy is passed on unattenuated by inviscid nonlinear interactions (e.g., vortex stretching, $\vec{\omega} \cdot \nabla \vec{u}$ where $\vec{\omega} = \nabla \times \vec{u}$ is the vorticity vector and $\vec{u}$ is the velocity vector) to smaller scales where it is ultimately dissipated by viscosity to heat, in which case we can write:

$$\frac{u^3}{L} = \varepsilon = \text{dissipation rate of turbulent kinetic energy} \quad (2.1)$$

a classic scaling estimate for the dissipation rate.

What is the smallest length scale of turbulence? It is set by the kinematic viscosity, $\nu$, and the turbulent kinetic energy transfer rate = turbulent kinetic energy dissipation rate $= \varepsilon$ in equilibrium flows. Applying dimensional analysis we find:

$$\begin{align*}
\ell & \sim \varepsilon^a \nu^b \\
[L] & = \left[ \frac{L^2}{T^3} \right] \\
\therefore \text{ for } [L] & \quad 1 = 2a + 2b \\
\therefore \text{ for } [T] & \quad 0 = 3a + b \\
1 & = -4a \quad \Rightarrow \quad a = -\frac{1}{4} \quad \Rightarrow \quad b = \frac{3}{4}
\end{align*}$$

$$\therefore \ell = \varepsilon^{-1/4} \nu^{3/4} = \left[ \frac{\nu^3}{\varepsilon} \right]^{1/4} = \eta$$

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Where η is known as the Kolmogorov length scale - the smallest length scale of a turbulent eddy.

What is the ratio of the largest length scale, $L$, to the smallest, $\eta$?

$$\frac{L}{\eta} \sim \frac{L}{\left(\frac{L^3}{\nu^3} \right)^{1/4}} \sim \frac{L u^{3/4}}{\nu^{3/4} L^{1/4}} \sim \left(\frac{L u}{\nu}\right)^{3/4} \sim Re_L^{3/4}$$

E.g., to computationally resolve a turbulent flow (direct numerical simulation) we must resolve $\eta \Rightarrow \Delta x \sim \eta$

where $\Delta x$ is the computational grid spacing, over the whole flow domain $\Rightarrow L$.

Hence the number of grid points required in any one coordinate direction to fully resolve the flow is $N = \frac{L}{\Delta x} \sim \frac{L}{\eta} \sim Re_L^{3/4}$.

Therefore in 3-D we would need $O(N^3) \Rightarrow Re_L^{9/4}$ points.

In Cayuga Lake $L \sim 1000m \quad u \sim 10^{-1} \text{ m/s} \quad \nu \sim 10^{-6} \text{ m/s}$.

Therefore $Re_L \sim 10^8 \Rightarrow N \sim 10^{18}$ or 1 billion billion points! That is order 1 billion Gigabytes of memory just to store your data at one time step, or 1 million Terabytes or 1 thousand Petabytes or what is known as an Exabyte. Not happening!

Today (2013) the typical largest simulations are run on grids of $4096^3$ points, or $O(10^{11})$ points, and this is rare!

Hence the only way to determine exactly how even a moderate Reynolds number flow behaves is to make measurements!

And instead of looking at all this data in an instantaneous sense we take a stochastic approach.

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2.1.2 Reynolds’ Decomposition

Let

\[ u = \overline{u} + u' \]  

(2.2)

where \( \overline{u} \) is the temporal mean component and \( u' \) is the fluctuating component. This decomposition is known as the Reynolds’ decomposition after Osborne Reynolds, who first proposed it as a statistical approach to aid in the study of turbulence.

The definition of \( \overline{u} \) is

\[ \overline{u} = \frac{1}{T} \int_0^T u \, dt \]  

(2.3)

Therefore

\[ \overline{u}' = 0 \]

\[ \overline{u} = \overline{\overline{u}} \]

\[ \overline{u}u' = 0 \]

But

\[ \overline{u}u' \neq 0 \]

O.K., to get a feel for the analysis of a turbulent signal let’s consider some data.

The signal below is the axial-component of velocity measured downstream from a turbulent round jet on the jet axis. The signal was collected with a one-component laser Doppler velocimeter (LDV), which basically measures the Doppler shift of light due to the motion of small seed particles in the direction of a light ray. This is somewhat analogous to an acoustic Doppler velocimeter (ADV), which we will discuss in detail.

The signal was sampled at \( f_s = 100 \) Hz for 194.56 seconds. Only 5 seconds of the signal is shown along with the \( \overline{u} = 0.1502 \) m/s (dashed line). Following Reynolds’ decomposition for this signal we can determine \( \overline{u'^2} = 0.0029 \) m\(^2\)/s\(^2\). The square root of this value, \( \sqrt{u'^2} = 0.0542 \) m/s, is often reported as a measure of the mean turbulence intensity level for a single velocity component. For those of you who recall your statistics this is essentially

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the standard deviation hence we expect about 63% of the velocity fluctuations to be less than 5.42 cm/s and 95% of the velocity fluctuations to be less than 10.8 cm/s if the statistics are normally distributed – which is a good assumption for the turbulent round jet. Looking at the figure we see this looks about right for the limited portion of the data we can see. In fact, let’s verify the assumption of a normally distributed data, how? Look at the histogram!

Now, let’s look at the spectrum (determined by averaging records of length of 512 points)

Note for completeness I have plotted the entire spectrum \( S_{uu} \) where the negative frequencies are on the right half of the plot (e.g., between 50 and 100 Hz - since the plot is log-log the plot does not appear symmetric but the data is!). Normally we would only plot \( G_{uu} \) (the positive frequencies) so the symmetric negative frequency data would not be shown.

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Of course, just to be confusing, it is entirely common to plot just the positive frequencies of $S_{uu}$ (i.e., without the factor of two on the amplitudes of $G_{uu}$) – hence you must be careful to understand what other researchers have done when you are looking at spectra! We see a well developed -5/3 region, known as the *inertial subrange*, which connects the low frequency region where the turbulence is produced to the high frequency region where the turbulence is dissipated. Now our data set does not really indicate the turbulence dissipation region as either instrument noise contaminates the signal at the highest frequencies - burying the actual turbulence signal which is decaying, or we have not sampled fast enough to resolve the dissipative frequencies. Looking at our data it appears to be a bit of both as there is evidence of the spectra beginning to flatten out at the highest frequencies, an indication of noise starting to dominate. Note that the integral of the spectrum yields the variance ($0.0029$ $m^2/s^2$)
The autocorrelation function (averaging records of length of 512 points) is shown below and on the top of the next page. We see that for the first 0.1 s or so the signal is strongly correlated to itself and this correlation then rapidly decays. After this, the signal slowly decays to a weakly negative correlation (second plot calculated with just 2 ensembles - \( N/2=9728 \) points, or 97.28 second length records). This all suggests that the turbulence forgets its history on the order of 0.1 - 1 second and behaves essentially randomly after this point. In fact, we define the autocorrelation time scale as the integral of the autocorrelation function. Integrating the second figure we find a time scale of 0.31 s. This is important information if one is interested in designing an efficient system for determining the mean or other moments as data that is truly independent with respect to each point will converge toward the true statistical value the most rapidly hence we are often interested in knowing the autocorrelation time scale.