2.3 The Turbulent Flat Plate Boundary Layer

The turbulent flat plate boundary layer (BL) is a particular case of the general class of flows known as boundary layer flows. The presence of a boundary requires a particular set of conditions be met there (generally the no-slip condition on the boundary-parallel components of velocity and that the component of flow normal to the boundary be zero, a kinematic constraint). The scale of boundary flows ranges from the wall boundary layer in a capillary tube up to the atmospheric boundary layer where the planet’s atmospheric circulation meets the earth. As we will show in a moment, boundary layer flows, in the absence of other forcing, are self-similar flows – meaning if we account for the effects of scale, all boundary layers look pretty much the same and have a unified non-dimensional form that handles variations in flow conditions and those due to spatial evolution.

Let’s consider a classic analytic model of a turbulent boundary layer that is relevant to all aspects of fluid mechanics – the flat plate boundary layer

When fluid flows over a boundary momentum is transferred from the fluid to the boundary, how does this occur? We can think of the momentum diffusing from the fluid to the
boundary (where it is converted to heat). The flowing fluid has a certain momentum

\[ \text{momentum} = \text{mass} \times \text{velocity} \quad \left[ \frac{ML}{T} \right] \]  

(2.17)

Where [M] signifies units of mass, [L] signifies units of length, and [T] signifies units of time. Since momentum is being lost to heat at the boundary we anticipate a flux of momentum toward the boundary

momentum flux = momentum per unit area per unit time = \frac{\text{mass} \times \text{velocity}}{\text{area} \times \text{time}} \quad \left[ \frac{M}{LT^2} \right] 

(2.18)

The velocity gradient at the boundary will induce a shear stress on the interface

\[ \text{shear stress} = \frac{\text{Force}}{\text{area}} \quad \left[ \frac{M}{L^2T} \right] \]

Now, as the flow continues downstream more and more momentum has been lost to the boundary resulting in a larger and larger momentum deficit at the boundary. The result is a boundary layer that grows with distance downstream as shown in the above picture.

Let’s consider our typical boundary layer (shown below)

The mass flow rate, \( \dot{m} \), on the left-hand-side is

\[ \dot{m}_{AB} = \rho A_{AB} \overline{u}_{AB} \]

where \( A_{AB} \) is the area of the control volume surface at the location \( A - B \). The mass flow rate on the right-hand-side is

\[ \dot{m}_{CD} = \rho A_{CD} \overline{u}_{CD} \]

conservation of mass (we invoke our incompressible flow assumption) tells us that

\[ \rho A_{AB} \overline{u}_{AB} = \rho A_{CD} \overline{u}_{CD} \]
But what do we know about $\overline{u}_{AB}$ relative to $\overline{u}_{CD}$? Since there is a stress at the boundary (which does work on the flow)

$$\overline{u}_{CD} < \overline{u}_{AB} \quad \Rightarrow \quad A_{CD} > A_{AB}$$

So our picture really looks like this

And for $A_{CD} > A_{AB}$ there must be some $\overline{w}$ along $A_{BC'}$, therefore

$$A_{AB}\overline{u}_{AB} = A_{C'D}\overline{u}_{C'D} + A_{BC'}\overline{w}.$$ 

### 2.3.1 Dimensional Analysis

As we stated earlier, boundary layers are self-similar meaning that we should be able to collapse all boundary layers in some non-dimensional sense.

We can define the shear velocity (often referred to as the friction velocity), $u^*$ as

$$u^* = \sqrt{\frac{\tau_w}{\rho}}$$ \hspace{1cm} (2.19)

where $\tau_w$ is the wall stress.

For a flat-plate boundary layer we can assume the plate is smooth and hence we ignore wall roughness $\Rightarrow$ our only parameters are the mean velocity, $\overline{u}$, the viscosity, $\nu$, and the shear stress, $u^*$. Dimensional analysis shows that we can make two non-dimensional groups from these three variables and they are

$$u^+ = \frac{\overline{u}(z)}{u^*}$$ \hspace{1cm} (2.20)

$$z^+ = \frac{u^* z}{\nu}$$ \hspace{1cm} (2.21)
where we recognize the later has the form of a Reynolds number (the product of velocity and length scales divided by the kinematic viscosity).

### 2.3.2 The Problem

We want to solve the equations of motion (Navier-Stokes equations) for the velocity profile in the fluid.

Simplifying assumptions:

- \( \rho \) is a constant
- Uniform flow in \( x \) direction \( \Rightarrow \) solution independent of \( y \) position \( (\partial / \partial y = 0) \) (and it can be shown that \( \overline{v} = 0 \))
- We’re far from the point where the fluid first touches the plate, hence the boundary layer is well developed, therefore it is only growing very slowly in the \( x \) direction \( \Rightarrow \partial / \partial x \ll \partial / \partial z \Rightarrow \) we will assume \( \partial / \partial x = 0 \) (which it can be shown leads to \( \overline{w} = 0 \))
- Steady state \( \Rightarrow \partial / \partial t = 0 \)

**Boundary Conditions**

- \( \overline{u}(z = 0) = \overline{u}(0) = 0 \) no-slip condition
- \( \tau_{xz}(z = 0) = \tau_{zx}(z = 0) = \tau_w \)

**The Solution**

From the assumptions we have \( \overline{u} = f(z) \) only. It can be shown that the equation for streamwise momentum (\( x \) component of Reynolds Averaged Navier-Stokes equation –
Eq. 2.8) reduces to (since $\partial/\partial x = \partial/\partial y = \overline{v} = \overline{w} = 0$)

$$0 = \frac{\partial}{\partial z} \left( \mu \frac{\partial \overline{u}}{\partial z} - \rho u'w' \right)$$

which, since our variables depend on $z$ only, becomes

$$0 = \frac{d}{dz} \left( \mu \frac{d \overline{u}}{dz} - \rho u'w' \right)$$

Integrating with respect to $z$

$$\mu \frac{d \overline{u}}{dz} - \rho u'w' = \text{a constant}$$

At $z = 0 \Rightarrow \overline{u}w' = 0$ but $\overline{u}/\partial z \neq 0$. We know that at the wall

$$\mu \left. \frac{\partial \overline{u}}{\partial z} \right|_{z=0} = \tau_w$$

Therefore we wish to solve

$$\tau_w = \mu \frac{d \overline{u}}{dz} - \rho u'w'$$

with the boundary condition $\overline{u}(0) = 0$.

As experimentalists we appear to be in pretty good shape as we can in fact measure all of the variables in Eq 2.24. However, analytically it turns out there is a problem. The original equations have more unknowns than equations. The standard solution is to express the Reynolds stress in terms of the other quantities. This is most simply done by analogy to the viscous stress using what is known as the eddy viscosity model

$$-\rho u'w' = \mu_t \frac{d \overline{u}}{dz} \Rightarrow -u'w' = \nu_t \frac{d \overline{u}}{dz}$$

where $\mu_t$ and $\nu_t$ are known as eddy viscosities (dynamic and kinematic) - which we can think of as an enhanced viscosity (just like $\nu$) due to the turbulence. Now, how do we handle the eddy viscosity? We model that – lets choose the classic model, the Prandtl mixing length model. Prandtl (1925) conjectured that it was not very realistic to assume a constant effective turbulent viscosity near a boundary as the size of the largest eddies depended on the distance from the boundary. Thus he developed a model based on a
mixing length, $\ell_m$, that was proportional to the distance from the boundary (e.g., $\ell_m \propto z$). Dimensionally, $\nu_t \propto [U][L]$ and Prandtl suggested appropriate velocity and length scales are $U$ and $\ell_m$ where $U$ is the mean velocity at the distance $\ell_m$ from the boundary. In a boundary layer $U$ can be approximated as $U = \ell_m |\partial \overline{u}/\partial z|$. What about $\ell_m$? The classic, and most theoretically based approach is to assume that the proportionality to the distance to the wall can be replaced with the Kolmogorov constant, $\kappa = 0.41$. Therefore

$$\ell_m = \kappa z$$

and putting it all together

$$-\overline{ww'} = \ell_m^2 \left( \frac{d\overline{u}}{dz} \right)^2$$

and Eq 2.24 becomes

$$\tau_w = \rho \nu \frac{d\overline{u}}{dz} + \rho \nu_t \frac{d\overline{\nu}}{dz}$$

$$= \rho \nu \frac{d\overline{u}}{dz} + \rho \kappa^2 z^2 \left( \frac{d\overline{u}}{dz} \right)^2$$

Now, how do we proceed? We’re engineers! Let’s use our engineering judgment to get the solution.

2.3.3 The Engineered Solution

For $z \leq d$

$z = d$ is some distance above the wall below which the length scales are too small for turbulence to play an appreciable role. Hence

$$\nu_t \approx 0 \quad \Rightarrow \quad \frac{d\overline{u}}{dz} = \frac{\tau_w}{\mu}$$

Integrating

$$\overline{u}(z) = \frac{\tau_w}{\mu} |z| + C_1$$

Invoking the boundary condition $\overline{u}(0) = 0$

$$\overline{u}(z) = \frac{\tau_w}{\mu} |z|$$
Multiplying by $\rho/\rho$

\[
\overline{u}(z) = \frac{\rho \tau_w}{\rho \mu} |z|
\]

But by definition

\[
u^* = \sqrt{\frac{\tau_w}{\rho}}\]
\[
\nu = \frac{\mu}{\rho}
\]

Therefore

\[
\overline{u}(z) = \frac{u^{*2}|z|}{\nu}
\]

or

\[
\overline{u}(z) = \frac{u^*|z|}{\nu}
\]

And recalling our wall scaled variables we have

\[
u^* = z^+
\]

For $z \geq d$

We now assume turbulence dominates over molecular processes. Hence

\[
\nu_t \gg \nu \implies \tau_w = \rho \kappa^2 z^2 \left( \frac{d\overline{u}}{dz} \right)^2
\]

or

\[
\left( \frac{d\overline{u}}{dz} \right)^2 = \frac{\tau_w}{\rho \kappa^2 z^2}
\]

Taking the square root of both sides

\[
\frac{d\overline{u}}{dz} = \frac{1}{\kappa z} \sqrt{\frac{\tau_w}{\rho}}
\]

Therefore

\[
\frac{d\overline{u}}{dz} = \frac{u^*}{\kappa z}
\]

Integrating

\[
\overline{u} = \frac{u^*}{\kappa} \ln z + C_2
\]

(2.30)
• What is the value of $C_2$?

• Where do the ‘viscous’ and ‘turbulent’ solutions match?

Unfortunately we can only form one equation by requiring the two solutions to match yet we have two unknowns.

⇒ Since we are engineers we look at the data to empirically get a value for one of the unknowns. It turns out that for smooth walls, loosely defined as (see discussion in Jiménez (2004). Turbulent Flows Over Rough Walls. Annu. Rev. Fluid Mech. 36, 173–96)

$$Re_{\text{roughness}} = \frac{u^* z_o}{\nu} \lesssim O(1)$$

(2.31)

where $z_o$ is known as the roughness height and is the mean height of the wall roughness elements, the matching point is at

$$z^+ = \frac{u^* z}{\nu} = 11.6$$

Therefore

$$\frac{u^* z}{\nu} = \frac{1}{\kappa} \ln |z| + \frac{C_2}{u^*} \quad \text{at } z = d = 11.6 \nu / u^*$$

And

$$11.6 = \frac{1}{\kappa} \ln \left( \frac{11.6 \nu}{u^*} \right) + \frac{C_2}{u^*}$$

and

$$\frac{C_2}{u^*} = 11.6 - \frac{1}{\kappa} \ln \left( \frac{11.6 \nu}{u^*} \right)$$

But from equation 2.30 we have

$$\bar{u} = \frac{1}{u^*} \ln z + \frac{C_2}{u^*}$$

Therefore

$$\bar{u} = \frac{1}{\kappa} \ln |z| + 11.6 - \frac{1}{\kappa} \ln \left( \frac{11.6 \nu}{u^*} \right)$$

Rearranging

$$\frac{\pi(z)}{u^*} = \frac{1}{\kappa} \ln \frac{u^* |z|}{\nu} - \frac{1}{\kappa} \ln 11.6 + 11.6$$
And recalling our wall-scaled variables

\[ u^+ = \frac{1}{\kappa} \ln z^+ + 5.5 \]  (2.32)

- Note that there is not a precise boundary separating the viscous sublayer from the log-layer and hence no actual intersection of the two velocity profile curves.

- A gradual transition occurs from the viscous dominated sublayer to the turbulence dominated log-layer from \( 3.5 < z^+ < 30 \).

- This velocity profile is considered to be universal for flow over smooth surfaces in the absence of heat transfer (or other stratifying mass flux) at the boundary.

- This universality allows flows with standard boundaries to be modeled much more simply as ‘wall functions’ can be used to impose the boundary layer structure on the flow without having to actually resolve the entire boundary layer (it takes a lot of grid points to get the boundary layer correct!).

Our engineered solution does very well away from the \( z = d = 11.6 \) region. How would we have approached an analytical solution given our assumption of a Prandtl mixing length model? We return to equation 2.29 and note the following:

- At \( z = 0 \) \( \tau(0) = 0 \)

- At \( z = 0 \) \( d\tau/dz > 0 \)

- For all \( z > 0 \) \( d\tau/dz \geq 0 \)

Now, putting things into wall coordinate form we first divide by \( \rho \):

\[ \frac{\tau_w}{\rho} = \frac{\mu}{\rho} \frac{d\tau}{dz} + \kappa^2 z^2 \left( \frac{d\tau}{dz} \right)^2 \]  (2.33)
By definition we have:

\[ u^+ = \frac{\overline{u}}{u_*} \]

Therefore

\[ u^* u^+ = \overline{u} \]

and

\[ \overline{u} = u^* u^+ \] (2.34)

Recalling our definition of \( z^+ \)

\[ z = \frac{\nu z^+}{u^*} \]

Now, differentiating equation 2.34 and using the chain-rule

\[ \frac{d\overline{u}}{dz} = \frac{d\overline{u}}{du^+} \frac{du^+}{dz^+} = u^* \frac{du^+}{dz^+} \left( \frac{u^*}{\nu} \right) \]

Therefore

\[ \frac{d\overline{u}}{dz} = \left( \frac{u^*^2}{\nu} \right) \frac{du^+}{dz^+} \]

Substituting this into equation 2.33 we have:

\[ u^*^2 = \nu \frac{d\overline{u}}{dz} + \kappa^2 z^+^2 \left( \frac{d\overline{u}}{dz} \right)^2 \]

\[ = u^*^2 \frac{du^+}{dz^+} + \kappa^2 z^+^2 u^*^2 \left( \frac{du^+}{dz^+} \right)^2 \]

Therefore

\[ \frac{du^+}{dz^+} + \kappa^2 z^+^2 \left( \frac{du^+}{dz^+} \right)^2 = 1 \] (2.35)

Now, solving for \( du^+/dz^+ \) and recalling that \( d\overline{u}/dz \geq 0 \) for all \( z \):

\[ \frac{du^+}{dz^+} = -\frac{1}{2\kappa^2 z^+^2} \left( \sqrt{1 + 4\kappa^2 z^+^2} - 1 \right) \] (2.36)

Equation 2.36 can be solved numerically. It turns out that it overestimates the mixing-length (\( \ell_m \)) near the wall. Van Driest developed a damping function that handles this problem. Van Driest proposed the following mixing-length parameterization:

\[ \ell_m = \kappa z \left[ 1 - \exp \left( -\frac{zu^*}{26\nu} \right) \right] \] (2.37)
If you re-work the derivation of equation 2.36 with the Van Driest mixing-length model you end up with

\[
\frac{du^+}{dz^+} = \frac{-1 + \sqrt{1 + 4\kappa^2 z^+} \left[ 1 - \exp \left( \frac{-z^+}{26} \right) \right]^2}{2\kappa^2 z^+ \left[ 1 - \exp \left( \frac{-z^+}{26} \right) \right]^2}
\]  

(2.38)

Which can also be solved numerically which I have done using a forward first-order difference scheme and is shown on the plots on pages 29 and 30.

On the following pages I have shown the direct numerical simulation (DNS) data of Philippe Spalart (1986). I have assumed a \( u^* = 0.01 \) m/s in dimensionalizing the data (consistent with a laboratory channel flow). Note that I could have chosen any measured \( u^* \) and reproduce a standard boundary layer.
The engineered solution results, the Van Driest model results along with Spalart’s DNS data are shown on the plot on page 30. It is clear that the engineered solution results work well in the limit but fail near $z = d = 11.6$. The Van Driest model works well throughout the domain.

### 2.3.4 The Multilayer Model

As we can see from the previous analysis one way of viewing a wall boundary layer is that there are multiple layers, each with their own dominant physical processes. These layers all have traditional names. So far we have focused our discussion on the inner layer - the region where the length scale is based on the viscous length scale ($z^+$). The inner layer is further broken down into the viscous wall region and the log region where the log region is the region of log scaling, typically taken to be ($z^+ > 30$), and the viscous
wall region is the region where viscous processes are important \((z^+ < 30)\). The viscous wall region is typically divided into two regions – the viscous sublayer, where \(U^+ = z^+\) is an excellent approximation (typically taken to be the region as \(z^+ < 3.5\)), and the buffer region where the transition from the fully viscous to the fully turbulent scaling occurs \((3.5 < z^+ < 30)\). The figure below (taken from Crimaldi, J.P. (1998) *Turbulence Structure of Velocity and Scalar Fields over a Bed of Model Bivalves*. Ph.D Dissertation, Stanford University) shows the typical nomenclature. Note we have focused our conversation on the inner layer. In the outer layer region what is known as the defect layer exists, which is discussed in detail in the Ligrani article handed out (Ligrani, P.M. (1989). Structure of turbulent boundary layers. In *Turbulence Phenomenon and Modeling, Encyclopedia of Fluid Mechanics*, ed. N.P. Cheremisinoff, Gulf Pub. Co.).

![Diagram of Turbulence Structure](image)

### 2.3.5 Turbulence Structure

The nondimensional streamwise turbulent fluctuations \((\sqrt{u'^2}/u^*)\) increase monotonically toward the wall until approximately \(z^+ = 15\) where they have a nondimensional peak of about 2.8 (exact position and magnitude are weakly a function of Reynolds number),
the profile then decreases monotonically to zero at \( z^+ = 0 \) where the no-slip boundary condition requires the fluctuations to be identically zero (see Ligrani Figure 13, Cowen and Monismith Fig 8).

The nondimensional vertical turbulent fluctuations \( \sqrt{w'^2/u^*} \) increase monotonically from zero away from the wall until approximately \( z^+ = 80 \) where they have a broad flat peak before decaying monotonically to the outer layer (see Ligrani Figure 18, Cowen and Monismith Fig. 8). The dominant Reynolds stress term is, nondimensionally, \( -\overline{u'w'}/u^{*2} \) which increases with distance from the wall to a broad peak near the region that \( \sqrt{w'/u^*} \) peaks. Note the nondimensional peak is slightly less than 1 as we are normalizing by the shear velocity (see Ligrani Figure 16, Cowen and Monismith Fig. 9b).

Let’s look at the Reynolds stress a bit more closely. Consider the component \( \overline{u'w'} \). Recall that this is the turbulent transport of momentum term. Physically the wall is a sink for momentum and we expect that the turbulence is trying to transport high momentum fluid toward the wall (and conversely low momentum fluid away from the wall). Given that \( \partial \tau / \partial z > 0 \), a negative \( w' \) fluctuation will carry high momentum fluid toward the wall and hence \( u' > 0 \). Thus in this region we expect \( \overline{u'w'} < 0 \). An easy way to get a sense for the Reynolds stress behavior is to use a scatter plot – a plot of \( u' \) vs \( w' \) in the case just described. Here are scatter plots from ADV data collected in a channel flow in the DeFrees Lab wide open channel flume. Working from left to right the elevation above the bed was \( z = 19.47 \text{ cm} \), \( z = 1.81 \text{ cm} \), and \( z = 0.1 \text{ cm} \).

What do we see? Starting away from the bed \((z = 19.47 \text{ cm})\) – the left-most plot – we