A $k-\varepsilon$ turbulence model based on the scales of vertical shear and stem wakes valid for emergent and submerged vegetated flows

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(Received ?? and in revised form ??)

Flow and transport through aquatic vegetation is characterized by a wide range of length scales: water depth ($H$), stem diameter ($d$), the inverse of the plant frontal area per unit volume ($a^{-1}$), and the scale(s) over which $a$ varies (the most easily identified is the plant height, $h$). Turbulence is generated both at the scale(s) of the mean vertical shear, set by the vertical distance(s) over which $a$ varies, and at the scale(s) of the stem wakes, set by $d$. While turbulence from each of these sources is dissipated through the energy cascade, some shear-scale turbulence bypasses the lower wave numbers as shear-scale eddies do work against the form drag of the plant stems, converting shear-scale turbulence into wake-scale turbulence. We have developed a $k-\varepsilon$ model that accounts for all of these energy pathways. The model is calibrated against laboratory data from beds of rigid cylinders under emergent and submerged conditions. The new model outperforms existing models, none of which include the $d$ scale, both in the emergent case, where existing models break down entirely, and in the submerged case, where existing models fail to predict the strong dependence of turbulent kinetic energy on $d$. The success of the new model supports our understanding of the turbulent kinetic energy budget in flow through vegetation. The new model was developed with applications in real aquatic vegetation in mind, and may be easily incorporated into larger hydrodynamic solvers for field applications.

Key Words: Turbulence modelling, Turbulent mixing, Geophysical and geological flows

1. Introduction

Aquatic vegetation influences flow and transport in wetlands, rivers, lakes, estuaries, and the coastal ocean. Flow through aquatic vegetation is characterized by several distinct length scales. Scales that are easily identified include the water depth, $H$, and a range of stem diameters, $d$. A less obvious length scale is the inverse of the plant frontal area per unit volume, $a^{-1}$. If there are $N$ plants in volume $V$, where $V$ includes both fluid and plants, and plant $i$ has area $A_i$ perpendicular to the mean flow, then the frontal area per unit volume, or frontal area density, is defined as

$$a = \frac{1}{V} \sum_{i=1}^{N} A_i.$$  \hspace{1cm} (1.1)

An illustration of $a$ is provided in figure 1 for rigid cylinders (often used to model aquatic vegetation in a laboratory setting). For many real plants, $a$ varies strongly in the vertical
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Figure 1. Illustration of $N = 5$ cylinders within a volume, $V$, showing the cylinder frontal areas, $A$, which are in the plane perpendicular to the mean velocity, $U$. In this case, $a = NA/V$.

direction. Thus, another important set of length scales in canopy flow is the range of scales over which $a(z)$ varies, where $z$ is the elevation above the bed.

Investigations into flow through aquatic vegetation have tended to focus either on emergent vegetation for which mean vertical shear is negligible and turbulence in plant wakes is the primary mechanism of mixing (e.g. Tanino & Nepf 2008b), or on deeply submerged vegetation in which a drag discontinuity at the top of the plant canopy leads to formation of a mixing layer, producing turbulence at the scale of the plant height that is destroyed in plant wakes (e.g. Ghisalberti & Nepf 2004). This division into emergent and submerged cases can work nicely for plants that resemble tall cylinders, such as reeds or sea grasses, but many leafy emergent plants have sufficiently non-uniform frontal area profiles that turbulence generated by vertical shear can be significant as well as wake turbulence.

The motivation for this work was our effort to predict velocities, turbulent diffusivities, and dispersion in Eurasian water milfoil (*Myriophyllum spicatum*), a fresh water plant that is native to Eurasia and Africa but rapidly spreading across North America as an invasive species. Milfoil is characterized by a highly non-uniform frontal area density profile, as shown in figure 2. Working with an emergent canopy of milfoil in a laboratory flume, Tinoco (2008) measured vertical profiles of velocity, turbulent kinetic energy (TKE), Reynolds stress, and dissipation using particle imaging velocimetry techniques.

Using the milfoil data, Tinoco (2008) tested a simple model developed by Lightbody & Nepf (2006), who were able to predict velocity profiles and longitudinal dispersion in a *Spartina alterniflora* wetland from a balance between drag and pressure gradients alone, i.e. neglecting vertical shear. In the case of milfoil, however, vertical shear was too significant to ignore, and the Lightbody & Nepf model performed poorly. We concluded that a model that predicts velocity profiles in a wide variety of real vegetation must include a parameterization for vertical shear. Given the scales of natural aquatic systems, a Reynolds-averaged Navier–Stokes (RANS) approach is appropriate, but because in real vegetation, $a = a(z)$, and thus vertical shear may vary over multiple, ill-defined length scales, a mixing length type model would be limited in applicability. Hence, a two-equation model such as $k$–$\varepsilon$, in which the turbulent length scale can adapt naturally to $a(z)$, is our simplest choice.

We originally hoped that the $k$–$\varepsilon$ model of L´opez & Garc´ıa (2001) could be adapted for
use in real vegetation. López & García modified the standard $k-\varepsilon$ model for application in aquatic plant canopies by adding an extra production term in the TKE and dissipation equations to account for the extra TKE produced in the plant wakes. To accommodate the moderate Reynolds numbers that milfoil experiences in nature, we modified the López & García model to account separately for viscous and pressure drag as described in King et al. (2009), and while with enough tuning, we were able to obtain decent fits to the mean velocity profiles measured in the laboratory milfoil canopies, we were not able to predict TKE or dissipation. Measurements revealed that dissipation scaled to a large degree with the effective stem diameter of the plants, defined as $d(z) = n/a(z)$, where $n$ is the number of stems per unit horizontal area; and dissipation scaled to a lesser degree with the multiple scales of the mean shear (R. O. Tinoco, personal communication). This led us to notice that the López & García model and other existing $k-\varepsilon$ models for flow through plant canopies (aquatic or terrestrial) do not anywhere incorporate stem diameter. Furthermore, we noticed that nearly all experiments used to calibrate such models for use in aquatic canopies have been conducted in beds of rigid cylinders having diameter $d = 6.4$ mm (1/4 in). In the well-studied case of densely packed emergent rigid cylinders in a laboratory setting, where turbulence is known to scale with $d$, these existing $k-\varepsilon$ models break down entirely.

To enable prediction of flow and transport in vegetated lakes, rivers, wetlands, and coastal areas, there is clearly a need for a two-equation model that incorporates the wake scale as well as the scale of the vertical shear. Such a model would be appropriate in vegetation with highly non-uniform $a(z)$ as well as reed-like vegetation, and would transition smoothly between emergent and submerged vegetation, such as in tidal flows. We develop such a model in the following sections.

After a discussion of the governing equations in §2 and an overview of the basic physical processes present in flow through aquatic vegetation in §3, we introduce the new $k-\varepsilon$ model in §4. While the model was developed with real plant canopies in mind, it is here calibrated against new laboratory data from flow through two arrays of rigid cylinders, one with cylinders of diameter $d = 3.2$ mm, and the other with cylinders of diameter $d = 25.4$ mm; each array is subjected to both emergent and submerged conditions. The
model is validated against existing laboratory data from beds of submerged rigid cylinders having diameter \( d = 6.4 \text{ mm} \). The experiments are discussed in §5, the calibration in §6, and the validation in §7. Model performance is compared to the performance of two existing \( k-\varepsilon \) models for flow through vegetation: those of López & García (2001) and of Katul et al. (2004). The new model outperforms both, not only in emergent cylinders where the other models break down entirely but also in submerged cylinders where the inclusion of stem-scale turbulence leads to better predictions of velocity profiles and dramatically better predictions of TKE.

The success of the new model provides insight into the physics of flow through aquatic vegetation, in particular the pathways of TKE generation and transfer between different wavenumbers. At this point, the model operates under the assumption of fully developed steady flow, but it would be simple to add terms for unsteadiness and advection. It is typical for 1D vertical mixing models such as this one to be incorporated into three-dimensional hydrodynamic models, such as Si3D (Smith 2006; Rueda & Schladow 2002), POM (Blumberg & Mellor 1987), or ROMS (Song & Haidvogel 1994) and into coastal wave models such as COBRAS (Lin & Liu 1998). Other possible applications extend to air flow through terrestrial canopies and urban landscapes, hydraulic flow around offshore structures, and some industrial flows.

2. Governing Equations

If the system of interest is a wetland, lake, river, estuary, or the coastal ocean (or in the case of air flow, a forest, field, or city), it is not computationally practical to resolve the flow field at the scale of individual plant stems. Hence, the approach of RANS modelers since Wilson & Shaw (1977) has been to horizontally average the governing equations over a scale large enough to smooth over heterogeneity due to the plant canopy structure. Raupach & Shaw (1982), building on the work of Wilson & Shaw, formalized two alternative spatial averaging procedures that are now widely known as schemes I and II. These two schemes were refined by Finnigan (1985) and by Raupach, Coppin & Legg (1986), who replaced the original horizontal average with a more general volume average. In scheme I, the equations are averaged over thin horizontal slabs large enough to smooth over heterogeneity due to turbulence as well as canopy structure but thin enough to preserve vertical gradients. In scheme II, a time average is employed to smooth over variations due to turbulence so that the spatial averaging volume need be extensive enough only to smooth over heterogeneity due to the canopy. Wilson & Shaw asserted that schemes I and II were essentially the same, but Raupach & Shaw later showed that they lead to different sets of averaged equations. Because it is rarely possible in laboratory or field settings to average over a large enough horizontal slab to smooth over heterogeneity due to turbulence, the equations developed under scheme II are more appropriate for comparison with experimental measurements.

The governing equations we review here are based on scheme II, in which a scalar field \( \psi \) (such as the streamwise velocity component, pressure, etc.) is decomposed into a time average, indicated by an overbar, and a fluctuation from that time average, indicated by a single prime:

\[
\psi = \overline{\psi} + \psi'.
\]

The time average is further decomposed into a spatial average, indicated by angle brackets, and the deviation from that spatial average, indicated by double primes:

\[
\overline{\psi} = \langle \psi \rangle + \psi''.
\]
The angle brackets indicate averaging over the fluid domain (i.e. excluding plant parts) within a thin horizontal slab large enough to smooth over heterogeneity due to the plant canopy but thin enough to preserve vertical gradients. For simplicity, let us consider steady, uniform, open-channel flow where \( x \) is the downstream coordinate and \( z \) is the elevation above the bed. Let \( u \) and \( w \) represent the instantaneous downstream and vertical velocity components, respectively.

### 2.1. Momentum Equation

Averaging the \( x \)-momentum equation for steady, uniform, open-channel flow under scheme II results in

\[
0 = gS + \frac{\partial \tau_{xz}}{\partial z} - f \tag{2.3}
\]

where \( g \) is the acceleration of gravity, \( S \) is the bed slope (equal to the surface slope), \( f \) is the drag force per unit fluid mass, and the mean shear stress, \( \tau_{xz} \), defined by

\[
\tau_{xz} \equiv \nu \frac{\partial \langle u \rangle}{\partial z} - \langle u'w' \rangle - \langle u''w'' \rangle, \tag{2.4}
\]

is the sum of the viscous stress, \( \nu \partial \langle u \rangle / \partial z \) (where \( \nu \) is the kinematic viscosity), the familiar Reynolds stress, \( -\langle u'w' \rangle \), and a dispersive stress, \( -\langle u''w'' \rangle \), that arises due to heterogeneity of the time-averaged velocity within the plant canopy. The drag term is composed of two components, \( f = f_p + f_\nu \) where the form (or pressure) drag force per unit fluid mass, \( f_p \), is given by

\[
f_p \equiv \frac{1}{\rho} \left\langle \frac{\partial p''}{\partial x} \right\rangle \tag{2.5}
\]

where \( \rho \) is the fluid density and \( p \) is pressure; and the viscous drag force per unit fluid mass, \( f_\nu \), is given by

\[
f_\nu \equiv -\nu \left\langle \nabla^2 \pi'' \right\rangle. \tag{2.6}
\]

Since Raupach & Shaw (1982), models have been based on the assumption that the dispersive stress, \( -\langle u''w'' \rangle \), is negligibly small compared to the Reynolds stress, \( -\langle u'w' \rangle \). There is plenty of experimental evidence that the dispersive stress is negligible above the plant canopy (e.g. Raupach et al. 1986; Poggi, Katul & Albertson 2004a), but within the plant canopy, Poggi et al. and also Bohm, Finnigan & Raupach (2000) measured large dispersive stresses. Comparing a range of canopy densities at Reynolds numbers over 100 000 (based on the depth average velocity and the water depth), Poggi et al. found that for canopies of density \( ah > 0.1 \), the dispersive stress is everywhere less than 10% of the Reynolds stress, but for sparse canopies, the dispersive stress can be on the order of the Reynolds stress. This observation is consistent with measurements by Raupach et al. and Bohm et al. In this paper, we assume that the dispersive stress is negligible in dense canopies at slightly lower Reynolds number, but this should be verified in the future.

### 2.2. Kinetic Energy Equations

The total kinetic energy, \( 1/2 \langle u_iu_i \rangle \), may be decomposed into mean kinetic energy (MKE), dispersive kinetic energy (DKE), and turbulent kinetic energy (TKE) as follows:

\[
\frac{1}{2} \langle u_iu_i \rangle = \frac{1}{2} \langle u_iu_i \rangle + \frac{1}{2} \langle \pi''_i \pi''_i \rangle + \frac{1}{2} \langle u''_i u''_i \rangle \tag{2.7}
\]

\[
\text{MKE} \quad \text{DKE} \quad \text{TKE}
\]
2.2.1. Turbulent Kinetic Energy (TKE)

The governing equation for TKE in steady, uniform, open channel flow is

\[
\frac{\partial T_z}{\partial z} = P_s + P_w - \varepsilon. \tag{2.8}
\]

where \( T_z \) is the transport term, \( P_s \) and \( P_w \) are production terms, and \( \varepsilon \) is the dissipation term. In flow through plant canopies, there are two mechanisms that produce TKE, represented by the two production terms. As in open water, TKE is produced by work of the Reynolds stress against the mean velocity gradient at rate \( P_s \), defined by

\[
P_s \equiv - \langle u'w' \rangle \frac{\partial \langle \pi \rangle}{\partial z}. \tag{2.9}
\]

Additionally, TKE is produced in the wakes of plant stems at rate \( P_w \), defined by

\[
P_w \equiv - \langle u' \, u'' \, \partial \pi'' \rangle. \tag{2.10}
\]

Historically, \( P_s \) has been called ‘shear production’ and \( P_w \) has been called ‘wake production’, and we adopt this nomenclature for consistency with previous work, although it is somewhat misleading because both ‘wake’ and ‘shear’ production involve shear. The turbulent transport, \( T_z \), is defined by

\[
T_z \equiv \frac{1}{2} \langle u'w'w'' \rangle + \frac{1}{\rho} \langle p''w'' \rangle - \nu \left\langle u' \left( \frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x_i} \right) \right\rangle + \frac{1}{2} \left\langle u' \, u'' \, \pi'' \right\rangle, \tag{2.11}
\]

and is identical to the turbulent transport in open water with the exception of the final term, a dispersive transport term.

Just as in the open water equations, the rate of dissipation of TKE to heat, or ‘dissipation’ for short, is defined by

\[
\varepsilon \equiv \nu \left\langle \frac{\partial u'}{\partial x_j} \left( \frac{\partial u'}{\partial x_j} + \frac{\partial w'}{\partial x_i} \right) \right\rangle. \tag{2.12}
\]

Note that we have separated the viscous transport from the dissipation of TKE to heat whereas Raupach & Shaw and others lump these terms together.

2.2.2. Dispersive Kinetic Energy (DKE)

The governing equation for DKE in steady, uniform flow is

\[
\frac{\partial \tilde{T}_z}{\partial z} = \tilde{P}_s + \tilde{P}_p - \tilde{\varepsilon} - \tilde{P}_w \tag{2.13}
\]

where \( \tilde{P}_s \), defined as

\[
\tilde{P}_s \equiv - \langle \pi'' \rangle \frac{\partial \langle \pi \rangle}{\partial z}, \tag{2.14}
\]

is production of DKE through work of the dispersive flux against the mean velocity gradient; \( \tilde{P}_p \), defined as

\[
\tilde{P}_p \equiv \langle \pi \rangle \frac{1}{\rho} \left\langle \frac{\partial p''}{\partial x_j} \right\rangle = \langle \pi \rangle f_p, \tag{2.15}
\]

is production of DKE through work of the mean velocity against the form drag; \( \tilde{T}_z \), defined as

\[
\tilde{T}_z \equiv \frac{1}{2} \langle u'' \, u'' \, \pi'' \rangle + \frac{1}{\rho} \langle p'' \, \pi'' \rangle - \nu \left\langle u'' \left( \frac{\partial \pi''}{\partial z} + \frac{\partial \pi'''}{\partial x_i} \right) \right\rangle + \frac{1}{2} \left\langle u'' \, u'' \, \pi''' \right\rangle, \tag{2.16}
\]
is the transport of DKE; and $\tilde{\varepsilon}$, defined as

$$ \tilde{\varepsilon} \equiv \nu \left( \frac{\partial \pi''_i}{\partial x_j} \left( \frac{\partial \pi''_i}{\partial x_j} + \frac{\partial \pi''_j}{\partial x_i} \right) \right), $$

(2.17)

is the rate of dissipation of DKE to heat. Notice that dissipation to heat is not the only sink for DKE: DKE is also converted into TKE through the wake production term, $P_w$, which was defined in (2.10). Also notice that viscous drag does not contribute to production of DKE.

In §2.1, we discuss evidence that the dispersive momentum flux, $\langle \pi'' \pi'' \rangle$, is small throughout and above dense canopies ($ah > 0.1$). Raupach & Shaw (1982) further assume that all dispersive fluxes are small. Under this assumption, the governing equation for DKE (2.13) simplifies to

$$ \tilde{P}_p = \tilde{\varepsilon} + P_w $$

(2.18)

López & García (2001) point out that there are two limiting cases for the DKE budget based on the relative scale of the plant stems ($d$) and the Kolmogorov microscale ($\eta$). Note that these limiting cases apply only when the dispersive fluxes are, indeed, negligible:

(a) When $d \gg \eta$, dissipation of DKE is negligible, i.e. $\tilde{\varepsilon} \approx 0$, and the DKE budget simplifies to

$$ P_w \approx \tilde{P}_p = \langle \pi \rangle f_p. $$

(2.19)

(b) When $d < \eta$, DKE produced by the work of the mean flow against plant stems is immediately dissipated to heat, i.e. $\tilde{P}_p \approx \tilde{\varepsilon}$, and thus the DKE budget simplifies to

$$ P_w \approx 0. $$

(2.20)

The relative scale of $d$ and $\eta$ is determined by the stem Reynolds number, $Re_d \equiv Ud/\nu$. At high $Re_d$, $d \gg \eta$, and at small $Re_d$, $d < \eta$. In either of these limiting cases, we obtain a simple closure for wake production, $P_w$, which appears in the governing equation for TKE, (2.8). If dispersive fluxes cannot be neglected, which appears to be the case in sparse canopies (i.e. canopies for which $ah < 0.1$), the closure of the TKE equation is not so simple. Our model, like existing RANS models for flow through vegetation, is based on the assumptions that dispersive fluxes are negligible and that $d \gg \eta$ so that (2.19) holds.

3. Physical Processes

We have developed our $k-\varepsilon$ model for applications in real aquatic vegetation, which is often characterized by a frontal area density profile, $a(z)$, that varies strongly in the vertical direction. However, the structure of the model is motivated by the physical processes observed in two very simple laboratory canopies. The simplest case for which $a(z)$ varies in the vertical direction is a model canopy of submerged rigid cylinders, illustrated in figure 3(a). In submerged cylinders, $a(z)$ is a step function, equal to a constant, $a_0$, within the canopy and zero above the canopy, as illustrated in figure 3(b). There is an easily identifiable length scale over which $a(z)$ varies – the height of the cylinders, $h$. Thus, in submerged cylinders, there are four relevant length scales: the cylinder height, $h$, the water depth, $H$, the cylinder diameter, $d$, and the frontal area density, $a_0^{-1}$.

An even simpler case is a model canopy of emergent cylinders, illustrated in figure 3(c). In this case, the cylinders reach (or protrude above) the water surface, so there are only three relevant length scales: $H$, $d$, and $a_0^{-1}$. Provided that the cylinder array is not too
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Figure 3. Illustrations of submerged (a) and emergent (c) cylinders, and plots of the frontal area density profile for submerged (b) and emergent (d) cylinders.

sparse, canopy drag dominates vertical shear and there are no vertical gradients, so we may neglect $H$, leaving only two relevant length scales: $d$ and $a_0^{-1}$. In this section, we first consider the physics of flow through dense canopies of emergent cylinders before moving on to consider the additional processes present in flow through submerged cylinders. The processes that enhance mixing in submerged cylinders are the same processes that enhance mixing in real plants that have more interesting frontal area profiles.

3.1. Emergent Cylinders

In canopies of sufficiently dense emergent cylinders, there are no vertical gradients, i.e. $\partial/\partial z = 0$. Thus, the momentum equation (2.3) simplifies to:

$$0 = gS - f. \quad (3.1)$$

It is not clear exactly where the transition between ‘sparse’ and ‘dense’ occurs for emergent cylinders, but extrapolating from the work of Luhar, Rominger & Nepf (2008) in submerged vegetation, we may estimate that vertical shear is negligible in emergent vegetation for $C_D a H / (1 - \phi) \gtrsim 0.1$ where $C_D$ is the drag coefficient and $\phi$ is the volume fraction occupied by the cylinders.

The drag force per unit fluid mass follows the quadratic drag law

$$f = \frac{1}{1 - \phi} \frac{1}{2} C_D a |U| U \quad (3.2)$$

where we have introduced the shorthand $U \equiv \langle \overline{u} \rangle$ for the mean velocity. It is possible to solve (3.1) and (3.2) analytically for the mean velocity, yielding

$$U = \pm \sqrt{\frac{2gS(1 - \phi)}{C_D a}}. \quad (3.3)$$
In the absence of vertical gradients, the TKE equation (2.8) simplifies to
\[ P_w = \epsilon. \]  
(3.4)

Tanino & Nepf (2008b) argue that in the wakes of emergent plants, dissipation scales with TKE and with plant diameter as follows:
\[ \epsilon \sim \frac{k^{3/2}}{d} \]  
(3.5)

provided that the inter-stem spacing is at least twice the stem diameter. Assuming that \( \eta \ll d \), we argued in §2.2.2 that \( P_w \approx U f_p \). Following Tanino & Nepf (2008b), for convenience, we may write separate quadratic drag laws for the pressure drag and the viscous drag, as follows:
\[ f_p = \frac{1}{1 - \phi} \frac{1}{2} C_{dp} \rho a |U|^2 |U| \]  
(3.6)
\[ f_v = \frac{1}{1 - \phi} \frac{1}{2} C_{dv} a |U| U. \]  
(3.7)

where we have decomposed the drag coefficient \( C_D = C_{dp} + C_{dv} \) into pressure and viscous contributions, respectively. Combining (2.19) and (3.6), and allowing for some inefficiency in conversion of DKE to TKE, we arrive at
\[ P_w \sim \frac{1}{1 - \phi} \frac{1}{2} C_{dp} a |U|^3. \]  
(3.8)

Solving (3.4), (3.5), and (3.8), for TKE, Tanino & Nepf (2008b) found that in emergent vegetation,
\[ k = \gamma U^2 \left( \frac{1}{1 - \phi} \frac{1}{2} C_{dp} a d \right)^{2/3} \]  
(3.9)

where we have introduced the shorthand \( k = \frac{1}{2} \langle u_i' u_i' \rangle \) for TKE and \( \gamma \) is an empirical constant. Tanino & Nepf (2008b) measured that \( \gamma = 1.21 \) in rigid, emergent cylinders. In general (even in plants with more complicated geometries), we expect \( \gamma \) to be order one provided that we can identify a characteristic diameter for the dissipation scaling.

In summary, in steady, fully developed, open channel flow through emergent cylinders, at high Reynolds number and with a high enough cylinder density, there are no vertical gradients, and we have the analytical solutions for the mean velocity and TKE, given in (3.3) and (3.9), respectively. These solutions are based on the free surface slope, the canopy drag coefficient, the frontal area density, and the cylinder diameter. Provided that the cylinders are sufficiently dense, there is only one identifiable dimensionless independent variable for the emergent problem: \( C_{dp} a d / (1 - \phi) \). This parameter is related to the volume fraction occupied by the cylinders (for cylinders, the volume fraction is \( \phi = \pi / 4 a d \)).

### 3.2. Submerged Cylinders

In submerged cylinders, vertical gradients are very important. Provided that dispersive fluxes and the viscous stress are small, the momentum equation (2.3) simplifies to
\[ 0 = gS - \frac{\partial \langle w^2 \rangle}{\partial z} f. \]  
(3.10)

The relative magnitude of the momentum transport and the drag force determines the qualitative nature of the flow. Arguing that the momentum transport scales as \( \partial \langle w^2 \rangle / \partial z \propto U^2 / h \), and employing the quadratic drag law (3.2) with a small plant
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volume fraction $\phi \approx 0$, Luhar et al. (2008) identify the dimensionless parameter $C_{Dahl}$, which is the ratio of drag to momentum transport. They observe that in sparse canopies (where $C_{Dahl} \lesssim 0.04$), momentum transport dominates, and flow resembles a rough boundary layer, while in dense canopies (where $C_{Dahl} \gtrsim 0.1$), drag dominates, and a mixing layer forms with the inflection point in mean velocity just below the top of the canopy. For non-negligible values of $\phi$, the dimensionless parameter representing the ratio of drag to momentum transport is $C_{Dahl}/(1 - \phi)$.

The TKE budget in a submerged canopy is governed by the full equation (2.8). There are two distinct mechanisms of TKE production, represented by the terms $P_s$ and $P_w$, and it is significant that the TKE produced by these two mechanisms has different characteristic length scales as this alters the pathways of dissipation. Vertical shear, and the turbulence generated by classic shear production, $P_s$, scales with the distance(s) over which $a(z)$ varies (in the case of rigid cylinders, this scale is $h$). TKE generated in the wakes of plant stems at rate $P_w$, on the other hand, scales with the characteristic diameter of the plant stems, $d$. These distinct scales are clearly visible in the energy spectra of both terrestrial and aquatic vegetation, e.g. see figure 7 of this paper and figure 5 of Poggi et al. (2004b). Shaw & Segner (1985) coined the terms ‘shear kinetic energy’ (or SKE) and ‘wake kinetic energy’ (or WKE) for the components of TKE produced by mean vertical shear and in the wakes of plant stems, respectively.

There is an additional important mechanism that is not explicit in the governing equation for TKE (2.8): energy is converted from SKE to WKE in the wakes of plant stems. Just as the mean flow does work against the mean plant drag to produce WKE and heat, large scale turbulent eddies do work against the fluctuating plant drag, converting SKE to WKE and heat. Assuming that DKE is negligible, that $d >> \eta$, and that the instantaneous plant drag may be parameterized by a quadratic law similarly to the mean drag, Finnigan (2000) writes that the rate (per unit mass) at which TKE is converted to WKE and heat in the wakes of plants is

$$W = \frac{1}{1 - \phi} \frac{1}{2} C_{Dp} a |(\bar{u}|u_i u_i) - U^3|$$

(3.11)

Note that we have added the factor $(1 - \phi)^{-1}$ to Finnigan’s expression to allow for canopies of significant volume fraction. $W$ is the work done by the instantaneous flow field against the instantaneous drag minus the work done by the mean flow field against the mean drag, which is the work done by the turbulence. Using a binomial expansion to approximate (3.11) to second order results in

$$W \approx \frac{1}{1 - \phi} \frac{1}{2} C_{Dp} a |U_{\frac{1}{2}} (\bar{u}_i u_i)|.$$  

(3.12)

Let us call $W$ the ‘spectral shortcut’ since it diverts SKE (at the scale of vertical shear) to WKE (at the smaller scale of the plant wakes).

Both SKE and WKE are dissipated via the turbulent eddy cascade as some of the SKE is diverted to WKE via the spectral shortcut. We may partition the total dissipation $\varepsilon$ into the components $\varepsilon_s$ and $\varepsilon_w$, representing the dissipation of SKE and WKE, respectively, through the energy cascade. A flow chart illustrating the energy pathways in submerged vegetation is given in figure 4. A similar flow chart was first published in Shaw & Segner (1985). Note that Wilson (1988) referred to this chart, and developed a Reynolds stress type model consistent with the energy pathways it illustrates, but fell short of including $k_w$ in his model equations.

In summary, in submerged vegetation, there are three identifiable dimensionless parameters governing the shape of the velocity, TKE, and Reynolds stress profiles. The
submergence ratio, $H/h$, governs the transition between deeply submerged and fully emergent vegetation. The parameter $C_{Dah}/(1 - \phi)$, identified by Luhar et al. (2008) as the ratio of drag force and vertical momentum transport, influences the shape of the velocity profile, in particular determining whether it is analogous to a rough boundary layer or a mixing layer. Finally, the parameter $C_{Dpd}/(1 - \phi)$, which is related to the plant volume fraction and governs the turbulence intensity in emergent vegetation, also has some influence in submerged vegetation through the scaling of WKE dissipation with stem diameter $d$.

4. Proposed Model

In this section, we propose a $k-\varepsilon$ model that naturally transitions from deeply submerged to emergent vegetation and is predictive over a wide range of all three dimensionless parameters: $H/h$, $C_{Dah}/(1 - \phi)$, and $C_{Dpd}/(1 - \phi)$. Existing $k-\varepsilon$ models for flow through aquatic vegetation do not include stem diameter, and thus miss the effects of $C_{Dpd}/(1 - \phi)$. Inclusion of $C_{Dpd}/(1 - \phi)$ is the main distinguishing feature of the new model. Our approach is to break TKE into its two components: SKE and WKE, and to treat dissipation of these two components separately. At this point, the model is for fully developed steady flow, and is appropriate for dense canopies (having $ah > 0.1$) and sufficiently high Reynolds number that dispersive fluxes and the viscous stress are negligible and that $d >> \eta$.

4.1. Momentum Equation

Our model momentum equation is identical to the momentum equation for existing $k-\varepsilon$ models such as those discussed in López & García (2001), Katul et al. (2004), and others. Assuming that dispersive fluxes and the viscous stress are negligible, the governing equation for momentum is given by (3.10). As in the standard model (Launder & Spalding 1974), we adopt the gradient diffusion hypothesis to model the Reynolds stress as follows:

$$\langle u'w' \rangle = -\nu_T \frac{\partial U}{\partial z}$$  (4.1)
where $\nu_T$ is the eddy viscosity, and arrive at the following model momentum equation:

$$0 = gS + \frac{\partial}{\partial z} \left( \nu_T \frac{\partial U}{\partial z} \right) - f$$

(4.2)

The quadratic drag law, given in (3.2), is used to parameterize the drag, $f$. We discuss the parameterization of eddy viscosity in §4.3.

### 4.2. TKE and Dissipation

We split TKE into two components, SKE (denoted $k_s$) and WKE (denoted $k_w$), so that total TKE is given by

$$k = k_s + k_w.$$ 

(4.3)

We also split dissipation into two components: dissipation of SKE through the energy cascade (denoted by $\varepsilon_s$) and dissipation of WKE (denoted by $\varepsilon_w$) so that total dissipation ($\varepsilon$) is given by

$$\varepsilon = \varepsilon_s + \varepsilon_w.$$ 

(4.4)

The model equation for SKE is

$$0 = \frac{\partial}{\partial z} \left( \frac{\nu_T}{\sigma_k} \frac{\partial k_s}{\partial z} \right) + P_s - \varepsilon_s,$$ 

(4.5)

and the model equation for WKE is

$$0 = \frac{\partial}{\partial z} \left( \frac{\nu_T}{\sigma_k} \frac{\partial k_w}{\partial z} \right) + P_w + W - \varepsilon_w.$$ 

(4.6)

The model equation for dissipation of SKE through the energy cascade is

$$0 = \frac{\partial}{\partial z} \left( \frac{\nu_T}{\sigma_\varepsilon} \frac{\partial \varepsilon_s}{\partial z} \right) + \frac{\varepsilon_s}{k_s} \left( C_{\varepsilon 1} P_s - C_{\varepsilon 2} \varepsilon_s - C_{\varepsilon 5} W \right)$$

(4.10)

where $\sigma_\varepsilon = 1.3$, $C_{\varepsilon 1} = 1.44$, and $C_{\varepsilon 2} = 1.92$ are standard model constants. The SKE-dissipation equation differs from the standard model equation for dissipation only in its inclusion of $W$ and the new model constant $C_{\varepsilon 5}$. We have included $W$ but not $P_w$ or $\varepsilon_w$ in this dissipation equation under the assumption that dissipation of SKE will be largely unaffected by the rate of production or dissipation of the smaller scale WKE.

Following Tanino & Nepf (2008b), we use the following algebraic equation to model
dissipation of WKE:

\[
\varepsilon_w = C_{\varepsilon D} \frac{k^3}{d^{3/2}} \tag{4.11}
\]

where \( C_{\varepsilon D} \) is a new model constant.

### 4.3. Eddy Viscosity

The eddy viscosity is parameterized by

\[
\nu_T = C_\mu \frac{k_s^2}{\varepsilon_s} + C_\lambda \frac{k_w^2}{\varepsilon_w} \tag{4.12}
\]

where \( C_\mu = 0.09 \) is a standard model constant and \( C_\lambda \) is a new model constant. Here we have assumed that SKE and WKE contribute to momentum diffusion in an additive way (analogous to turbulent and molecular diffusion of scalars). In fact, we expect SKE and WKE to interact once they are generated, but it seems reasonable to assume that these contributions to momentum diffusion are additive to first order, and making this assumption ensures that in the absence of vegetation, where \( k_w \) goes to zero, the eddy viscosity collapses to the standard parameterization. Since the standard constant \( C_\mu \) is tuned for mixing layers, plane jets, and boundary layers, we have introduced a different constant, \( C_\lambda \), for momentum transport via wake turbulence.

### 4.4. Limiting Cases

In the limiting case of no plants, all TKE is SKE, \( W = 0 \), and the model equations collapse to the standard model equations. In the limiting case of dense emergent vegetation, where vertical gradients are small, the momentum equation collapses to \( gS = f \), yielding the solution for mean velocity given in (3.3), and production, \( P_s \), is negligible, so no SKE is ever generated, and the model equation for WKE becomes \( P_w = \varepsilon_w \), in agreement with (3.4). For consistency with the solution for TKE given in (3.9), the model constants \( C_{\varepsilon D} \) and \( \beta_p \) must be related by

\[
C_{\varepsilon D} = \beta_p \gamma^{-3/2}. \tag{4.13}
\]

We use \( \gamma = 1.21 \) as found by Tanino & Nepf (2008b).

### 4.5. Numerical Implementation

Our numerical algorithm employs a finite volume spatial discretization, illustrated in figure 5, and a Crank-Nicolson time iteration method (we add a time derivative and allow the equations to relax to steady state). \( \nu_T \) is evaluated at the nodes, and TKE and dissipation are evaluated at the faces so that \( \nu_T \) may easily be evaluated at the faces. The convergence criterion is that the larger of the source or sink terms comes within 0.1% of the smaller of the source or sink terms plus the transport term for all four of the transport equations (momentum, SKE, WKE, and dissipation of SKE).

### 4.6. Boundary Conditions

Our boundary conditions are consistent with the open water boundary conditions described in Burchard & Peterson (1999). We have chosen open water boundary conditions because in dense vegetation, the numerical solution is not sensitive to boundary conditions (i.e. we can get away with it), and the open water boundary conditions provide for a smooth transition between flow through vegetated canopies and open water flows, e.g. in the context of a three-dimensional hydrodynamic model used in field applications where vegetation may not cover the entire bed.
For the momentum boundary condition at the bed, the shear stress is specified as follows:

\[ \nu T \frac{\partial U}{\partial z} = u^2_{*,b} \text{ at } z = 0. \] (4.14)

The friction velocity, \( u_{*,b} \), is estimated from the mean velocity at node 1, \( U_1 \), using the quadratic drag law

\[ u^2_{*,b} = C_{D,b} U^2_1 \] (4.15)

where \( C_{D,b} \) is a drag coefficient. Assuming a logarithmic velocity profile near the bed, we may calculate \( C_{D,b} \) from the roughness height, \( z_0 \), as follows:

\[ C_{D,b} = \left( \frac{1}{\kappa} \ln \frac{z_1}{z_0} \right)^{-2} \] (4.16)

where \( \kappa = 0.41 \) is the von Kármán constant and \( z_1 \) is the elevation of node 1.

For the momentum boundary condition at the free surface, the shear stress is specified as follows:

\[ \nu T \frac{\partial U}{\partial z} = u^2_{*,s} \text{ at } z = H. \] (4.17)

The friction velocity, \( u_{*,s} \), is estimated from the wind speed \( U_{10} \), measured at a standard 10 m above the water surface, using the quadratic drag law

\[ u^2_{*,s} = C_{D,s} U_{10}^2. \] (4.18)

For laboratory flows, we have set \( U_{10} = 0 \). Wüst & Lorke (2003) provide a nice overview of estimates for \( C_{D,s} \) in the field.

For SKE, we use the following Dirichlet boundary condition at the bed:

\[ k_s = u^2_{*,b} \sqrt{C_D} \text{ at } z = 0 \] (4.19)

and a zero stress boundary condition near the free surface:

\[ \frac{\nu T \partial k_s}{\sigma_k \partial z} = 0 \text{ at } z = z_N. \] (4.20)

For WKE, we use zero stress boundary conditions near the bed and near the free
\[ k-\varepsilon \text{ model for vegetated flows} \]

surface:

\[
\frac{\nu_T}{\sigma_k} \frac{\partial k_w}{\partial z} = 0 \text{ at } z = z_1 \text{ and } z = z_N. \tag{4.21}
\]

For dissipation of SKE, we use the following Dirichlet condition at the bed:

\[
\varepsilon_s = C_3^{3/4} \frac{k_s^{3/2}}{\kappa(z_1 + z_{0,b})} \text{ at } z = 0 \tag{4.22}
\]

and the following Neumann condition near the free surface:

\[
\frac{\nu_T}{\sigma_\varepsilon} \frac{\partial \varepsilon_s}{\partial z} = \frac{C_\mu}{\sigma_\varepsilon} \frac{k_s^2}{(H - z_N) + z_{0,s}} \text{ at } z = z_N \tag{4.23}
\]

where the surface roughness, \( z_{0,s} \), is zero in the case of no wind stress.

There is no need for a boundary condition for dissipation of WKE because \( \varepsilon_w \) is an algebraic function of \( k_w \).

5. Experiments

For model calibration, four experiments were conducted in an open channel flume in the DeFrees Hydraulics Laboratory at Cornell University. For model validation, we used twelve experiments described in Dunn, López & García (1996) that were conducted in the Hydrosystems Laboratory at the University of Illinois at Urbana-Champaign. Both sets of experiments were conducted in canopies of rigid cylinders. The experimental methods and results are described in this section.

5.1. Cornell University Experiments

5.1.1. Methods

The 4.50 m long, 0.600 m wide recirculating type open channel flume at Cornell was fitted with 3.60 m long, 0.600 m wide arrays of randomly located, rigid, vertically mounted acrylic cylinders. The random cylinder locations were chosen from a uniform distribution of possible locations constrained by the criteria that the center of cylinders must be a minimum of two diameters apart and that the surface of all cylinders must be located at minimum 3.2 mm (1/8 in) from the edge of a plate. The cylinders were inserted into holes CNC machined into 6.4 mm (1/4 in) thick PVC 0.600 m wide base plates with lengths of 1.20 m (4.00 ft). The random array pattern is periodic in the longitudinal direction with periodic length scale the array plate length, i.e. 1.20 m.

Two acrylic cylinder arrays were used, comprised of 0.200 m long cylinders, one with cylinders of 3.2 mm (1/8 in) diameter and the other with cylinders of 25.4 mm (1 in) diameter, each populated to a frontal area density of \( a = 4.00 \text{ m}^{-1} \) (1260 cylinders/m² and 158 cylinders/m², respectively). To allow for optical access to the flow, cylinders were not located in a full width 150 mm long gap that begins 2.80 m from the upstream end of the random array plates. For the experiments discussed here, measurements were taken at the upstream edge of the gap with two Nortek Vectrino acoustic Doppler velocimeters (ADVs) configured with the Vectrino+ firmware allowing sample rates to 200 Hz. Dye visualizations and ADV measurements confirmed that the gap was sufficiently narrow to have no discernible impact on the flow. Steady fully developed flow was established and verified via mid-depth ADV measurements along the length of the emergent cylinder array and by measurements of a fully developed linear Reynolds stress profile above the canopy for the submerged case.

For each of the two cylinder diameters, experiments were conducted under emergent
(\(H/h < 1\)) and submerged (\(H/h = 1.5\)) conditions. To account for high lateral variability in the flow due to the cylinder wakes, measurements were taken at seven transverse coordinates, \(y = \{90, 160, 230, 300, 370, 440, 510\}\) mm, where \(y\) is the distance from the left wall of the flume, looking downstream. At each transverse coordinate, measurements were taken at six depths for the emergent cases and twelve or thirteen depths for the submerged cases. The ADV was oriented to look either downward or upstream depending on the measurement location (the upstream-looking orientation was necessary to obtain measurements within 50 mm of the free surface). Records were collected with ADVs sampling at 200 Hz for 10 min at each measurement location.

To obtain estimates of \(U\), \(\langle u'w' \rangle\), and \(k = 1/2\langle u'_i u'_i \rangle\), traditional Reynolds decomposition (in time) and averaging were performed first, then spatial averages were taken over the seven lateral measurement locations at a given elevation. Because measurements were taken in a gap that was free of cylinders, measured mean longitudinal velocities and Reynolds stresses were multiplied by \((1 - \phi)^{-1}\) to correct for the expansion of flow into the gap. In order to estimate the uncertainty due to lateral flow variability not captured by the seven measurement locations, a bootstrap sampling technique was applied (Efron & Tibshirani 1993): seven time-averaged statistics were sampled with replacement 1000 times from the original seven measured statistics, the mean was calculated for each sample of seven, and histograms of the means were used to obtain the upper and lower bounds of the 95% confidence interval. To compute power spectra, each 200 Hz velocity time series was time-averaged down to a sampling frequency of 50 Hz, then spectra were computed using ten sub-windows and averaged across the seven lateral sampling locations.

The pressure drag coefficient, \(C_{DP}\), was evaluated by fitting measurements of \(k\) and \(U\) to (3.9) in the emergent cases using \(\gamma = 1.21\) from Tanino & Nepf (2008b). Tanino & Nepf (2008a) found that \(C_{DP}\) is independent of Reynolds number, varying only with the cylinder volume fraction \(\phi\), so we assume that \(C_{DP}\) in each submerged case is equivalent to \(C_{DP}\) in the emergent case with the corresponding \(\phi\). The viscous drag coefficient, \(C_{DV}\), was evaluated from the standard curve of drag coefficient vs. Reynolds number for a single cylinder, found in any introductory fluids text (e.g. Finnemore & Franzini 2002, p. 383), with one subtracted to account for the pressure drag on a single cylinder. We assume that the viscous drag is independent of \(\phi\).

The water surface slope, \(S\), was estimated from the velocity statistics. For the emergent cylinder experiments, \(S\) was estimated using the depth-average velocity, the total drag coefficient, and (3.3). For the submerged cylinder experiments, \(S\) was estimated from the balance of the momentum flux and the gravitational force in the open water region, \(gS = \partial \langle u'w' \rangle \partial z\), as follows. First, the Reynolds stress at the top of the canopy, \(\langle u'w' \rangle_h\), was estimated from a linear least squares fit to the Reynolds stress data above the canopy, assuming that \(\langle u'w' \rangle = 0\) at \(z = H\). Then, \(\langle u'w' \rangle_h\) was used to calculate the friction velocity scale \(u_* \equiv \langle u'w' \rangle_h^{1/2}\). Finally, the slope was estimated as \(S = u_*^2/g (H - h)^{-1}\).

Geometric parameters, velocity scales, drag coefficients, and Reynolds numbers for the Cornell University experiments are reported in table 1, and the dimensionless parameters governing the flow are reported in table 2.

5.1.2. Results

Power spectra are plotted in figure 6 for the emergent cylinder experiments (E1 and E2) and in figure 7 for the submerged cylinder experiments (S1 and S2). In the emergent cases, a clear peak is visible at the cylinder Strouhal frequency, and a \(-5/3\) slope is observed at higher frequencies, indicating that TKE is generated at a single length scale in the wakes of the cylinders. For experiment S1 (the submerged case with \(d = 0.32\) cm
Table 1. Geometric parameters, velocities, Reynolds numbers, and drag coefficients for the Cornell University experiments. $U_Q$ is the laterally- and depth-averaged mean velocity; $U_0 = U_Q$ for the emergent cylinder experiments (E1 and E2) and $U_0$ is the average of the three laterally-averaged mean velocities measured nearest the bed for the submerged cylinder experiments (S1 and S2); $U_h$ is the laterally-averaged mean velocity at the top of the cylinders; $Re_H = U_Q H/ν$; and $Re_d = U_0 d/ν$.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>$H/h$</th>
<th>$C_{D_p} H/(1 - φ)$</th>
<th>$C_{D_v} d/(1 - φ)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>E1</td>
<td>$\leq 1$</td>
<td>1.26</td>
<td>0.019</td>
</tr>
<tr>
<td>E2</td>
<td>$\leq 1$</td>
<td>0.75</td>
<td>0.119</td>
</tr>
<tr>
<td>S1</td>
<td>1.50</td>
<td>1.43</td>
<td>0.019</td>
</tr>
<tr>
<td>S2</td>
<td>1.53</td>
<td>0.90</td>
<td>0.119</td>
</tr>
</tbody>
</table>

Table 2. Dimensionless parameters for the Cornell University experiments. Note that $C_{D_p} H/(1 - φ)$ is reported in place of $C_{D_v} h/(1 - φ)$ for the emergent cylinder experiments (E1 and E2).

and $h = 29.6\text{ cm}$, a peak is observed at frequency $0.4U_h/h$, indicating that energy is injected into the flow through the shear instability that forms at the canopy top. This peak is present at all depths throughout and above the canopy. Additionally, a clear peak is visible at the cylinder Strouhal frequency within the canopy, indicating that both SKE and WKE make important contributions to TKE. For experiment S2 (the submerged case with $d = 2.54\text{ cm}$ and $h = 29.6\text{ cm}$), the cylinder height and cylinder diameter are close enough in scale that it is not possible to distinguish two distinct peaks, though this does not exclude the possibility that TKE is generated in the cylinder wakes as well as the shear layer.

Vertical profiles of mean velocity, Reynolds stress, and and TKE are plotted in figures 8 and 9 for the emergent and submerged cases, respectively. In the emergent cases, vertical gradients and Reynolds stress are negligible, as predicted for dense emergent canopies (note that $C_{D_p} H/(1 - φ) > 0.1$ for both E1 and E2), and the profiles of $k/U_0^2(1 - φ)^2/(2C_{D_v} d)^{2/3}$ collapse to a constant, in agreement with (3.9). In the submerged cases, we find very similar velocity and Reynolds stress profiles for S1 and S2, but the TKE level differs by an order of magnitude between S1 and S2 within the canopy.
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Figure 6. Power spectra for the emergent canopy experiments, E1 and E2: $S_{uu}$ (solid line), $S_{vv}$ (dashed line), and $S_{ww}$ (dash/dotted line). These spectra are computed from measurements taken at $z/H = 0.48$. A line having $-5/3$ slope and a vertical line at the cylinder Strouhal frequency, $0.2U_0/d$, are plotted as well.

5.2. University of Illinois Experiments

A complete description of the University of Illinois experiments may be found in Dunn et al. (1996). These experiments were carried out in a tilting flume that was 19.5 m long, 0.91 m wide, and 0.61 m deep. Plants were modeled using rigid wooden dowels having diameter $d = 0.635$ cm (1/4 in). The tops of the cylinders were at height $h = 11.8$ cm above the bed. Cylinders were not randomly distributed but arranged in the staggered patterns illustrated in figure 4.3 of Dunn et al. (1996). In all cases, the cylinders were submerged. The experiments were conducted under steady uniform flow conditions at very high Reynolds number (based on the depth average velocity and the cylinder diameter, above 55,000 for all experiments). Velocity profiles were measured using an ADV in a downward-looking orientation. For each experiment, velocity profiles were measured at four points in the horizontal plane chosen to capture the variability in the flow field due to the presence of the cylinders. Bed slope was measured directly, but where Reynolds stress measurements above the canopy top are available, we used estimates of $S$ based on a linear least squares fit to the Reynolds stress profile as described in §5.1.1 for the Cornell University submerged experiments. Following López & García (2001), we used a drag coefficient of $C_D = 1.13$ for all the experiments. Since the experiments were conducted at high Reynolds number, we assume that $C_{DW} = 0$ and $C_{DP} = C_D$.

The dimensionless parameters governing the flow in the Illinois experiments are reported in table 3. Reynolds number and bed slope are reported in table 5.3 of Dunn et al. (1996). Profiles of $U$, $\langle u'w' \rangle$, and the velocity variances $\langle u'^2 \rangle$, $\langle v'^2 \rangle$, and $\langle w'^2 \rangle$ that make up the TKE $k = 1/2(\langle u'^2 \rangle + \langle v'^2 \rangle + \langle w'^2 \rangle)$ were obtained from Appendix B of Dunn et al. (1996).

6. Model Calibration

The new model we introduced in §4 includes five empirical coefficients (in addition to the standard $k-\varepsilon$ model coefficients): $\beta_p$, $\beta_d$, $C_{\varepsilon 5}$, $C_{\varepsilon D}$, and $C_\lambda$. Scaling arguments suggest that $\beta_p$, $\beta_d$, and $C_{\varepsilon 5}$ are order one, and $\beta_p$ cannot exceed 1 because it is an energy conversion efficiency. We expect $C_\lambda$ to be of order similar to the standard model coefficient $C_\mu = 0.09$. $C_{\varepsilon D}$ is determined by $\beta_p$ and $\gamma$ according to (4.13), so there are
$k-\varepsilon$ model for vegetated flows

Figure 7. Power spectra for the submerged canopy experiments. For experiment S1, spectra are plotted at elevations $z/h = 0.25$ (solid line), $z/h = 0.61$ (dashed line), $z/h = 1.02$ (dash/dotted line), and $z/h = 1.37$ (dotted line). For experiment S2, spectra are plotted at elevations $z/h = 0.20$ (solid line), $z/h = 0.61$ (dashed line), $z/h = 1.02$ (dash/dotted line), and $z/h = 1.37$ (dotted line). A line having $-5/3$ slope and vertical lines at the cylinder Strouhal frequency, $0.2U_0/d$, and at the frequency $0.4U_h/h$ are plotted as well.

Figure 8. Vertical profiles of mean velocity, Reynolds stress, and TKE measured in the emergent canopy experiments E1 ($\circ$) and E2 ($\times$). Error bars indicate 95% confidence intervals obtained from the bootstrap mean of measurements from 7 lateral locations.
Figure 9. Vertical profiles of mean velocity, Reynolds stress, and TKE measured in the submerged canopy experiments S1 (○) and S2 (×). Error bars indicate 95% confidence intervals obtained from the bootstrap mean of measurements from 7 lateral locations.

Table 3. Dimensionless parameters for the University of Illinois (Dunn et al. 1996) experiments.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>$H/h$</th>
<th>$C_{pah}/(1 - \phi)$</th>
<th>$C_{DpD}/(1 - \phi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.85</td>
<td>0.146</td>
<td>0.0079</td>
</tr>
<tr>
<td>2</td>
<td>1.95</td>
<td>0.146</td>
<td>0.0079</td>
</tr>
<tr>
<td>3</td>
<td>1.40</td>
<td>0.146</td>
<td>0.0079</td>
</tr>
<tr>
<td>4</td>
<td>2.35</td>
<td>0.146</td>
<td>0.0079</td>
</tr>
<tr>
<td>5</td>
<td>1.73</td>
<td>0.146</td>
<td>0.0079</td>
</tr>
<tr>
<td>6</td>
<td>2.27</td>
<td>0.036</td>
<td>0.0020</td>
</tr>
<tr>
<td>7</td>
<td>1.56</td>
<td>0.036</td>
<td>0.0020</td>
</tr>
<tr>
<td>8</td>
<td>3.33</td>
<td>0.331</td>
<td>0.0179</td>
</tr>
<tr>
<td>9</td>
<td>1.82</td>
<td>0.331</td>
<td>0.0179</td>
</tr>
<tr>
<td>10</td>
<td>2.26</td>
<td>0.331</td>
<td>0.0179</td>
</tr>
<tr>
<td>11</td>
<td>2.65</td>
<td>0.082</td>
<td>0.0044</td>
</tr>
<tr>
<td>12</td>
<td>1.98</td>
<td>0.082</td>
<td>0.0044</td>
</tr>
</tbody>
</table>

four degrees of freedom in our calibration. We use the directly measured value $\gamma = 1.21$ from Tanino & Nepf (2008b).

We calibrated the model using the two submerged Cornell University experiments (S1, and S2). The model was executed for each experiment with coefficient $\beta_p$ varied between 0 and 1 with a resolution of 0.1, coefficients $\beta_d$, and $C_{\varepsilon 5}$ varied between 0 and 3 with a resolution of 0.1, and the coefficient $C_\lambda$ varied between 0 and 0.3 with a resolution of 0.001 to obtain a best fit in the mean square sense. Mean square error (model vs. data) was equally weighted for experiments S1 and S2 and for $U$, $\langle u'w' \rangle$ and $k$. The mean square error for each vertical profile was normalized by the variance of the data to capture the fraction of the variability explained by the model.

The coefficients giving the best fit to the Cornell University data are reported in Table 4. $\beta_p$, $\beta_d$, and $C_{\varepsilon D}$ are all within an order of magnitude of one, confirming the scaling arguments inherent in the model. $C_\lambda$ is an order of magnitude smaller than $C_{\mu}$, suggesting that wake-scale turbulence contributes less to vertical mixing than turbulence generated by vertical shear. Surprisingly, $C_{\varepsilon 5} = 0$, suggesting that the rate at which shear-scale turbulence is dissipated through the energy cascade is independent of the rate at which it is lost to wake-scale turbulence via form drag.
7. Model Performance and Validation

The performance of the new model was compared to the performance of the \( k-\varepsilon \) models described in López & García (2001), which we will call the ‘López–García model’, and in Katul et al. (2004), which we will call the ‘Katul model’. Both of these models include only one length scale of TKE and dissipation. The López–García model may be obtained from our model by summing the SKE and WKE equations, neglecting the effects of WKE on eddy viscosity, and neglecting the effects of \( W \) on dissipation, although the López–García model coefficients are different from ours. The Katul model is based on the assumption that WKE is quickly dissipated and thus negligible. Under this assumption, the model includes the conversion of SKE to WKE, \( W \), as a sink of TKE. However, the Katul model also includes production of WKE, \( P_W \), as a source of TKE, which seems to violate the assumption that WKE is negligible. These two models are discussed in further detail in Appendices A and B. Identical discretizations, boundary conditions, and convergence criteria were used for all three models.

The advantages of the new model are most striking for the emergent cylinder experiments (E1 and E2). The results for the three models are plotted along with the experimental data in figure 11, and the mean square errors are reported in table 5. While all three models accurately predict \( U \) and \( \langle u'w' \rangle \), only the new model predicts the dependence of \( k \) on \( C_D a d/(1 - \phi) \). The López–García and Katul models predict vertical gradients in \( k \) while the new model correctly predicts uniform \( k \) across the water depth. Note that the López–García model results are dependent on the convergence criterion, i.e. the model fails to converge in the emergent case (the López–García model has no convergence problems in the submerged case, for which it was developed).

As in the emergent cases, the advantages of the new model are most clear for \( k \) in the submerged cylinder experiments (S1 and S2). The results for the three models are
plotted along with the experimental data in figure 12, and the mean square errors are reported in table 5. Again, the absence of the parameter $C_{Dad}/(1 - \phi)$ from the López–García and Katul models results in their inability to predict the dependence of $k$ on the cylinder diameter as it is varied between experiments S1 and S2; the new model, in contrast, captures the $k$ profiles quite accurately. The new model is comparable to the López–García model and superior to the Katul model in capturing the $U$ and $\langle u'w' \rangle$ profiles.

The new model, along with the coefficients found by calibration against the Cornell University experiments, was validated against the twelve submerged cylinder experiments reported in Dunn et al. (1996). The López–García and Katul models were tested against this same data for comparison. The results are plotted in figure 13, and mean square errors (model vs. data) are reported in table 6. The new model is clearly superior to the other two models, especially in prediction of $k$, but also in prediction of $U$. The López–García model makes slightly better predictions of $\langle u'w' \rangle$, but all three models predict this quantity quite well. Note that in the few cases where the new model underperforms (Experiments 2, 8, 12), there is either evidence that the flow was not fully developed (see the concave $\langle u'w' \rangle$ profile in experiments 2 and 8) or no data above the plant canopy, and thus grounds for suspicion that the flow may not have been fully developed (Experiment 12).

8. Conclusions

To our knowledge, we have developed the first model for flow through vegetation (aquatic or terrestrial) to handle all of the energy pathways laid out by Shaw & Seginer (1985) as illustrated in figure 4. The model is of the $k$–$\varepsilon$ type, predicting vertical mixing, and may be easily incorporated into larger two-dimensional or three-dimensional hydrodynamic codes.

Previously developed RANS models have accounted for turbulence production at the scale of vertical shear and at the scale of plant stems, but have not allowed for different pathways of dissipation for these two scales of turbulence, such that $d$ is not featured...
Our model was designed to collapse properly to the algebraic solutions for $U$, $(\langle u'w' \rangle)/u^2_*$ and $k$ in the well-studied case of dense emergent rigid cylinders and calibrated to predict the dependence of $k$ on $d$ observed in the laboratory, which it does quite well. Our model also outperforms the López–García and Katul models, which do not include the $d$ scale, in predicting $U$ and $k$ in laboratory data from submerged cylinders published by Dunn et al. (1996).

The two major advantages of our model are its ability to transition smoothly between emergent and submerged cases, essential in tidal flows, and its applicability in the case of leafy emergent aquatic vegetation, in which turbulence generated by vertical shear and turbulence generated in the wakes of plant stems can be equally important. The model’s superior performance in the case of submerged cylinder canopies suggests that inclusion of stem-scale turbulent dissipation may improve predictions of velocity and TKE profiles even in deeply submerged or terrestrial canopies (of plants or of buildings).

The authors wish to thank Francisco Zarama and Rafael Tinoco for collecting the laboratory data discussed in §5. They also wish to thank Rafael Tinoco and Francisco Rueda for valuable conversations that inspired and informed this work. This material is based upon work supported by the National Science Foundation under grant CBET-
Figure 13. Model predictions compared to the University of Illinois experiments. Experimental data (●) is plotted at all four lateral measurement locations at each elevation. Results from the new model are plotted with a solid line, results from the López–García model are plotted with a dashed line, and results from the Katul model are plotted with a dashed/dotted line.
Table 5. Mean square errors (model vs. measured data) for the Cornell University experiments. For the emergent experiments (E1 and E2), each mean square error is normalized by the squared mean of the measured vertical profile; for the submerged experiments (S1 and S2), each mean square error is normalized by the variance of the measured vertical profile.

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<th>new model</th>
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Appendix A. López–García Model

The López–García model is described in detail in López & García (2001). The momentum equation is identical to (4.2). The TKE equation is given by

$$0 = \frac{\partial}{\partial z} \left( \nu_T \frac{\partial k}{\partial z} \right) + P_s + P_w - \varepsilon$$  \hspace{1cm} (A 1)

where $P_s$ and $P_w$ are modeled as in (4.7) and (4.8), respectively. Note that in López & García (2001), $\beta_p$ is called $C_{fk}$, and $C_{fk} = 1.0$. Note that if we sum the two TKE equations in the new model, (4.5) and (4.6), we arrive at (A 1), so the López–García equation for TKE is identical to ours; the differences are in the dissipation equation and the turbulent eddy viscosity. López–García model the dissipation using a single equation:

$$0 = \frac{\partial}{\partial z} \left( \nu_T \frac{\partial \varepsilon}{\partial z} \right) + \frac{\varepsilon}{k} \left( C_{c1} P_s - C_{c2} \varepsilon + C_{f\varepsilon} P_w \right)$$  \hspace{1cm} (A 2)

where $C_{f\varepsilon}$ is a model constant. As in the standard $k-\varepsilon$ model, the turbulent eddy viscosity is modeled as $\nu_T = C_{\mu} k^2 / \varepsilon$. Arguing that in the absence of mean shear (i.e. in dense, emergent vegetation), wake production must balance dissipation, López & García (2001) conclude that $C_{f\varepsilon} = (C_{c2} / C_{c1}) C_{fk} = 1.33$. While we agree that in the absence of mean
shear, wake production must balance dissipation, we disagree that (A 2) is the proper way to model dissipation in emergent vegetation, where dissipation scales with stem diameter, which is not included anywhere in (A 2).

<table>
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Table 6. Mean square errors (model vs. measured data) for the University of Illinois experiments, each normalized by the variance of the measured vertical profile.
Appendix B. Katul Model

The Katul model is described in detail in Katul et al. (2004). The momentum equation is identical to (4.2). The TKE equation is given by

\[ 0 = \frac{\partial}{\partial z} \left( \frac{\nu_T}{\sigma_k} \frac{\partial k}{\partial z} \right) + P_s + P_w - W - \varepsilon \]  

where \( P_s \), \( P_w \), and \( W \) are modeled as in (4.7), (4.8), and (4.9). The Katul model for TKE differs from the new model and from the López & García (2001) model by the inclusion of \( W \), the rate of conversion of SKE to WKE. The argument for including \( W \) as a sink for total TKE is that small-scale WKE is dissipated very quickly. We find it puzzling that under this assumption, the WKE production term, \( P_w \), is also included in the TKE equation. The dissipation rate, \( \varepsilon \), is modeled using

\[ 0 = \frac{\partial}{\partial z} \left( \frac{\nu_T}{\sigma_\varepsilon} \frac{\partial \varepsilon}{\partial z} \right) + \frac{\varepsilon}{k} (C_{\varepsilon 1} P_s - C_{\varepsilon 2} \varepsilon + C_{\varepsilon 4} P_w - C_{\varepsilon 5} W) \]  

where \( C_{\varepsilon 4} \) and \( C_{\varepsilon 5} \) are model constants. As in the standard \( k-\varepsilon \) model, the turbulent eddy viscosity is modeled as \( \nu_T = C_\mu k^2 / \varepsilon \). Katul et al. (2004) use different model constants for terrestrial and aquatic vegetation. For aquatic vegetation, they use \( \beta_p = 1.0 \), \( \beta_d = 4.0 \), \( C_{\varepsilon 4} = 1.5 \), and \( C_{\varepsilon 5} = 1.5 \).

REFERENCES


