Finite Control Volume Analysis

Application of Reynolds Transport Theorem

CEE 331
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Moving from a System to a Control Volume

- Mass
- Linear Momentum
- Moment of Momentum
- Energy

Putting it all together!
Conservation of Mass

B = Total amount of **mass** in the system

b = **mass** per unit mass = 1

\[
\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t} \int_{cv} \rho b \, dV + \int_{cs} \rho b \mathbf{V} \cdot \hat{n} \, dA \quad \text{cv equation}
\]

\[
\frac{DM_{sys}}{Dt} = \frac{\partial}{\partial t} \int_{cv} \rho \, dV + \int_{cs} \rho \mathbf{V} \cdot \hat{n} \, dA \quad \text{But } DM_{sys}/Dt = 0!
\]

\[
\int_{cs} \rho \mathbf{V} \cdot \hat{n} \, dA = - \frac{\partial}{\partial t} \int_{cv} \rho \, dV
\]

**Continuity Equation**

mass leaving - mass entering = - rate of increase of mass in cv
Conservation of Mass

\[
\int \rho \mathbf{V} \cdot \hat{n} \, dA = -\frac{\partial}{\partial t} \int \rho \, dV
\]

If mass in cv is constant

\[
\int \rho_1 \mathbf{V}_1 \cdot \hat{n}_1 \, dA + \int \rho_2 \mathbf{V}_2 \cdot \hat{n}_2 \, dA = 0
\]

Unit vector \( \hat{n} \) is normal to surface and pointed out of cv

We assumed uniform \( \rho \) on the control surface

\( \overline{V} \) is the spatially averaged velocity normal to the cs

\[
\overline{V} = \frac{\int_{cs} \mathbf{V} \cdot \hat{n} \, dA}{A}
\]
Continuity Equation for Constant Density and Uniform Velocity

\[ \int_{c_{s_1}} \rho_1 \mathbf{V}_1 \cdot \hat{n}_1 \, dA + \int_{c_{s_2}} \rho_2 \mathbf{V}_2 \cdot \hat{n}_2 \, dA = 0 \]
\[ -\rho_1 \mathbf{V}_1 A_1 + \rho_2 \mathbf{V}_2 A_2 = 0 \]

Density is constant across \(c_{s}\)

Density is the same at \(c_{s_1}\) and \(c_{s_2}\)

\[ \mathbf{V}_1 A_1 = \mathbf{V}_2 A_2 = Q \, [L^3/T] \]

Simple version of the continuity equation for conditions of constant density. It is understood that the velocities are either \underline{uniform} or \underline{spatially averaged}.
Example: Conservation of Mass?

The flow out of a reservoir is 2 L/s. The reservoir surface is 5 m x 5 m. How fast is the reservoir surface dropping?

\[
\int_{cs} \rho \mathbf{V} \cdot \mathbf{n} \, dA = - \frac{\partial}{\partial t} \int_{cv} \rho \, dV
\]

\[
\int_{cs} \mathbf{V} \cdot \mathbf{n} \, dA = - \frac{\partial V}{\partial t}
\]

\[
Q_{out} - Q_{in} = - \frac{dV}{dt}
\]

\[
Q_{out} = - \frac{A_{\text{res}}}{dt} \frac{dh}{dt} = - \frac{Q}{A_{\text{res}}}
\]

Constant density

Velocity of the reservoir surface

Example
Linear Momentum Equation

\[ \frac{DB_{\text{sys}}}{Dt} = \frac{\partial}{\partial t} \int_{cv} \rho b \, dV + \int_{cs} \rho b \mathbf{V} \cdot \hat{n} \, dA \quad \text{cv equation} \]

\[ \mathbf{B} = m \mathbf{V} \quad \text{momentum} \quad \mathbf{b} = \frac{m \mathbf{V}}{m} \quad \text{momentum/unit mass} \]

\[ \frac{Dm \mathbf{V}}{Dt} = \frac{\partial}{\partial t} \int_{cv} \rho \mathbf{V} \, dV + \int_{cs} \mathbf{V} \rho \mathbf{V} \cdot \hat{n} \, dA \]

\[ \frac{Dm \mathbf{V}}{Dt} = \int_{cs} \mathbf{V} \rho \mathbf{V} \cdot \hat{n} \, dA \quad \text{Steady state} \]

This is the “\( \text{ma} \)” side of the \( F = ma \) equation!
Linear Momentum Equation

\[
\frac{D m \mathbf{V}}{D t} = \int_{cs} \mathbf{V} \rho \mathbf{V} \cdot \hat{n} \, dA
\]

\[
\frac{D m \mathbf{V}}{D t} = \int_{cs_1} \mathbf{V}_1 \rho_1 \mathbf{V}_1 \cdot \hat{n}_1 \, dA + \int_{cs_2} \mathbf{V}_2 \rho_2 \mathbf{V}_2 \cdot \hat{n}_2 \, dA
\]

\[
\frac{D m \mathbf{V}}{D t} = - (\rho_1 V_1 A_1) \mathbf{V}_1 + (\rho_2 V_2 A_2) \mathbf{V}_2
\]

\[
\mathbf{M}_1 = - (\rho_1 V_1 A_1) \mathbf{V}_1 = - (\rho Q) \mathbf{V}_1
\]

\[
\mathbf{M}_2 = (\rho_2 V_2 A_2) \mathbf{V}_2 = (\rho Q) \mathbf{V}_2
\]

Assumptions

- Uniform density
- Uniform velocity
- \( V \perp A \)
- Steady
- \( V \) fluid velocity relative to cv

Vectors!!!
Steady Control Volume Form of Newton’s Second Law

\[ \sum \mathbf{F} = \frac{D (m \mathbf{V})}{D t} = \mathbf{M}_1 + \mathbf{M}_2 \]

What are the forces acting on the fluid in the control volume?

- Gravity
- Shear at the solid surfaces
- Pressure at the solid surfaces
- Pressure on the flow surfaces

\[ \sum \mathbf{F} = \mathbf{M}_1 + \mathbf{M}_2 \]

Why no shear on control surfaces? No velocity tangent to control surface
Resultant Force on the Solid Surfaces

- The shear forces on the walls and the pressure forces on the walls are generally the unknowns.
- Often the problem is to calculate the total force exerted by the fluid on the solid surfaces.
- The magnitude and direction of the force determines:
  - size of _______ needed to keep pipe in place
  - force on the vane of a pump or turbine...

\[ \sum \mathbf{F} = \mathbf{W} + \mathbf{F}_{p_1} + \mathbf{F}_{p_2} + \mathbf{F}_{ss} \]

\[ \mathbf{F}_{ss} = \mathbf{F}_{p\text{wall}} + \mathbf{F}_{\tau\text{wall}} \]

=force applied by solid surfaces
Linear Momentum Equation

\[ \sum \mathbf{F} = \mathbf{W} + \mathbf{F}_{p_1} + \mathbf{F}_{p_2} + \mathbf{F}_{ss} \]
\[ m \mathbf{a} = \mathbf{M}_1 + \mathbf{M}_2 \]
\[ \mathbf{M}_1 + \mathbf{M}_2 = \mathbf{W} + \mathbf{F}_{p_1} + \mathbf{F}_{p_2} + \mathbf{F}_{ss} \]

Forces by solid surfaces on fluid

The momentum vectors have the same direction as the velocity vectors

\[ \mathbf{M}_1 = - (\rho Q) \mathbf{V}_1 \]
\[ \mathbf{M}_2 = (\rho Q) \mathbf{V}_2 \]
Example: Reducing Elbow

Reducing elbow in vertical plane with water flow of 300 L/s. The volume of water in the elbow is 200 L. Energy loss is negligible.

Calculate the force of the elbow on the fluid.

\[ W = -\rho g \cdot \text{volume} = -1961 \text{ N} \uparrow \]

<table>
<thead>
<tr>
<th>section 1</th>
<th>section 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>D 50 cm</td>
<td>30 cm</td>
</tr>
<tr>
<td>A 0.196 m²</td>
<td>0.071 m²</td>
</tr>
<tr>
<td>V 1.53 m/s</td>
<td>4.23 m/s</td>
</tr>
<tr>
<td>p 150 kPa</td>
<td>?</td>
</tr>
<tr>
<td>M -459 N</td>
<td>1270 N</td>
</tr>
<tr>
<td>F_p 29,400 N</td>
<td>?←</td>
</tr>
</tbody>
</table>

\[ M_1 + M_2 = W + F_{p_1} + F_{p_2} + F_{ss} \]

Direction of \(V\) vectors
Example: What is $p_2$?

\[
\frac{p_1}{\gamma_1} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma_2} + z_2 + \frac{V_2^2}{2g}
\]

\[
p_2 = p_1 + \gamma \left[ z_1 - z_2 + \frac{V_1^2}{2g} - \frac{V_2^2}{2g} \right]
\]

\[
p_2 = \left(150 \times 10^3 \text{ Pa}\right) + \left(9810 \text{ N/m}^3\right) \left[0 - 1 \text{ m} + \frac{(1.53 \text{ m/s})^2 - (4.23 \text{ m/s})^2}{2 \left(9.8 \text{ m/s}^2\right)}\right]
\]

$P_2 = 132 \text{ kPa}$

$F_{p2} = 9400 \text{ N}$
Example: Reducing Elbow Horizontal Forces

\[
M_1 + M_2 = \mathcal{W} + F_{p_1} + F_{p_2} + F_{ss}
\]

\[
F_{ss} = M_1 + M_2 - \mathcal{W} - F_{p_1} - F_{p_2}
\]

\[
F_{ssx} = M_{1x} + M_{2x} - \mathcal{W}_x - F_{p_{1x}} - F_{p_{2x}}
\]

\[
F_{ssx} = M_{2x} - F_{p_{2x}}
\]

\[
F_{ssx} = (1270 \, N) - (-9400 \, N)
\]

\[
F_{ssx} = 10.7 \, kN \quad \text{Force of pipe on fluid}
\]

Fluid is pushing the pipe to the **left**
Example: Reducing Elbow Vertical Forces

\[ F_{ssz} = M_{1z} + M/_{2z} - \omega_z - F_{p1z} - F_{p2z} \]

\[ F_{ssz} = M_{1z} - \omega_z - F_{p1z} \]

\[ F_{ssz} = -459 \text{N} - (1,961 \text{N}) - (29,400 \text{N}) \]

\[ F_{ssz} = -27.9 \text{kN} \quad \text{28 kN acting downward on fluid} \]

Pipe wants to move up
Example: Fire nozzle

A small fire nozzle is used to create a powerful jet to reach far into a blaze. Estimate the force that the water exerts on the fire nozzle. The pressure at section 1 is 1000 kPa (gage). Ignore frictional losses in the nozzle.
Fire nozzle: Solution

Identify what you need to know

\( P_2, V_1, V_2, Q, M_1, M_2, F_{ss} \)

Determine what equations you will use

Bernoulli, continuity, momentum
Find the Velocities

\[
\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_f}{\gamma} + \frac{V_2^2}{2g}
\]

\[
\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{V_2^2}{2g}
\]

\[
p_1 = \frac{V_2^2}{\gamma} - \frac{V_1^2}{2g}
\]

\[
p_1 = \rho \frac{V_2^2}{2} \left( 1 - \left( \frac{D_2}{D_1} \right)^4 \right)
\]

continuity → \[
V_1 D_1^2 = V_2 D_2^2
\]

\[
V_2^2 \left( \frac{D_2}{D_1} \right)^4 = V_1^2
\]

\[
V_2 = \sqrt{\frac{2p_1}{\rho \left[ 1 - \left( \frac{D_2}{D_1} \right)^4 \right]}}
\]
Fire nozzle: Solution

<table>
<thead>
<tr>
<th>section 1</th>
<th>section 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>0.08</td>
</tr>
<tr>
<td>A</td>
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</tr>
<tr>
<td>P</td>
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<tr>
<td>V</td>
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</tr>
<tr>
<td>Fp</td>
<td>5027</td>
</tr>
<tr>
<td>M</td>
<td>-96.8</td>
</tr>
<tr>
<td>Fssx</td>
<td>-4132</td>
</tr>
<tr>
<td>Q</td>
<td>22.1</td>
</tr>
</tbody>
</table>

Which direction does the nozzle want to go? _____

Is $F_{ssx}$ the force that the firefighters need to brace against? **NO! Moments!**

force applied by nozzle on water

\[ F_{ssx} = M_{1x} + M_{2x} - \omega \times - F_{p1x} - F_{p2x} \]
Example: Momentum with Complex Geometry

Find $Q_2$, $Q_3$ and force on the wedge in a horizontal plane.

$Q_1 = 10 \text{ L/s} \quad V_1 = 20 \text{ m/s}$

$F_y = 0$

$\theta_1 = 10^\circ \quad \theta_2 = 130^\circ \quad \theta_3 = -50^\circ$

$\rho = 1000 \text{ kg/m}^3$

Unknown: $Q_2$, $Q_3$, $V_2$, $V_3$, $F_x$
5 Unknowns: Need 5 Equations

Identify the 5 equations!

Continuity \( Q_1 = Q_2 + Q_3 \)

Bernoulli (2x)

\[
\frac{p_1}{\gamma_1} + \frac{V_1^2}{2g} = \frac{p_2}{\gamma_2} + \frac{V_2^2}{2g}
\]

\( V_1 = V_2 \)

\( V_1 = V_3 \)

Momentum (in x and y)

\[
M_1 + M_2 + M_3 = W + F_{p_1} + F_{p_2} + F_{p_3} + F_{ss}
\]
Solve for $Q_2$ and $Q_3$

$M_1 + M_2 + M_3 = W + F_{p_1} + F_{p_2} + F_{p_3} + F_{ss}$  

atmospheric pressure

$F_{ssy} = 0 = M_{1y} + M_{2y} + M_{3y}$

$M_1 = -\left(\rho Q \right) V_1$

$0 = -\rho Q_1 V_1 \sin \theta_1 + \rho Q_2 V_2 \sin \theta_2 + \rho Q_3 V_3 \sin \theta_3$

$V \sin \theta = \text{Component of velocity in y direction}$

$Q_1 = Q_2 + Q_3$  

Mass conservation

$V_1 = V_2 = V_3$  

Negligible losses – apply Bernoulli
Solve for $Q_2$ and $Q_3$

\[ 0 = -\rho Q_1 I_1 \sin \theta_1 + \rho Q_2 I_2 \sin \theta_2 + \rho Q_3 I_3 \sin \theta_3 \]

Eliminate $Q_3$

\[ 0 = -Q_1 \sin \theta_1 + Q_2 \sin \theta_2 + Q_3 \sin \theta_3 \]

\[ Q_2 = Q_1 \frac{(-\sin \theta_1 + \sin \theta_3)}{(-\sin \theta_2 + \sin \theta_3)} \]

\[ Q_2 = Q_1 \frac{[-\sin(10) + \sin(-50)]}{[-\sin(130) + \sin(-50)]} \]

Why is $Q_2$ greater than $Q_3$?

\[ \dot{m}_1 V_{1y} = \dot{m}_2 V_{2y} + \dot{m}_3 V_{3y} \]

\[ Q_2 = 6.133 \text{ L/s} \]

\[ Q_3 = 3.867 \text{ L/s} \]
Solve for $F_{ssx}$

\[ F_{ssx} = M_{1x} + M_{2x} + M_{3x} \]

\[ F_{ssx} = -\rho Q_1 V_1 \cos \theta_1 + \rho Q_2 V_1 \cos \theta_2 + \rho Q_3 V_1 \cos \theta_3 \]

\[ F_{ssx} = \rho V_1 [ -Q_1 \cos \theta_1 + Q_2 \cos \theta_2 + Q_3 \cos \theta_3 ] \]

\[ F_{ssx} = (1000 \text{ kg/m}^3)(20 \text{ m/s}) \left[ -\left(0.01 \text{ m}^3/\text{s}\right) \cos(10) + \left(0.006133 \text{ m}^3/\text{s}\right) \cos(130) + \left(0.003867 \text{ m}^3/\text{s}\right) \cos(-50) \right] \]

\[ F_{ssx} = -226N \quad \text{Force of wedge on fluid} \]
Vector solution

\[ \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 = \mathbf{F}_{ss} \]

\[ |\mathbf{M}_1| = -\rho Q_1 V_1 = 200\text{N} \]

\[ |\mathbf{M}_2| = |\rho Q_2 V_2| = 122.66\text{N} \]

\[ |\mathbf{M}_3| = |\rho Q_3 V_3| = 77.34\text{N} \]

\[ Q_2 = 10 \text{ L/s} \]

\[ Q_2 = 6.133 \text{ L/s} \]

\[ Q_3 = 3.867 \text{ L/s} \]
Vector Addition

\[ \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 = \mathbf{F}_{ss} \]

Where is the line of action of \( \mathbf{F}_{ss} \)?
Moment of Momentum Equation

\[ \frac{D B_{\text{sys}}}{D t} = \frac{\partial}{\partial t} \int_{c_v} \rho b dV + \int_{c_s} \rho b V \cdot \hat{n} dA \quad \text{cv equation} \]

\[ B = m r \times V \quad \text{Moment of momentum} \]

\[ b = \frac{m r \times V}{m} \quad \text{Moment of momentum/unit mass} \]

\[ \frac{D (m r \times V)}{D t} = \frac{\partial}{\partial t} \int_{c_v} \rho r \times V dV + \int_{c_s} \rho (r \times V)(V \cdot \hat{n}) dA \]

\[ \sum T = \int_{c_s} \rho (r \times V)(V \cdot \hat{n}) dA \quad \text{Steady state} \]
Application to Turbomachinery

\[ T = \int_{cs} \rho (r \times V) (V \cdot \hat{n}) \, dA \]

\[ \int_{cs} \rho (V \cdot \hat{n}) \, dA = \rho Q \]

\[ T_z = \rho Q \left[ (r_2 \times V_2) - (r_1 \times V_1) \right] \]
Example: Sprinkler

Total flow is 1 L/s. Jet diameter is 0.5 cm. Friction exerts a torque of 0.1 N·m·s⁻² ω². θ = 30°. Find the speed of rotation. $V_t$ and $V_n$ are defined relative to control surfaces.
Example: Sprinkler

\[ 0.1\omega^2 + \rho Q r_2^2 \omega - \rho Q^2 r_2 \frac{2}{\pi d^2} \sin \theta = 0 \]

\[ a = 0.1 \text{Nms}^2 \]

\[ b = \rho Q r_2^2 \]

\[ b = (1000 \text{ kg/m}^3)(0.001 \text{ m}^3/\text{s})(0.1 \text{ m})^2 = 0.01 \text{ Nms} \]

\[ c = -\rho Q^2 r_2 \frac{2}{\pi d^2} \sin \theta \]

\[ c = -(1000 \text{ kg/m}^3)(0.001 \text{ m}^3/\text{s})^2(0.1 \text{m})(2\sin30)/3.14/(0.005 \text{ m})^2 \]

\[ c = -1.27 \text{ Nm} \]

What is \( \omega \) if there is no friction? \( \omega = 127/\text{s} \)

\( \omega = 3.5/\text{s} \)

What is \( V_t \) if there is no friction? \( 0 \)

\[ = 34 \text{ rpm} \]

\[ T = \rho Q r_2 V_{t_2} \]

Reflections
Energy Equation

\[
\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t} \int_{cv} \rho b dV + \int_{cs} \rho b V \cdot \hat{n} dA \quad \text{cv equation}
\]

\[
\frac{DE}{Dt} = \frac{\partial}{\partial t} \int_{cv} \rho edV + \int_{cs} \rho e V \cdot \hat{n} dA \quad \text{What is } \frac{DE}{Dt} \text{ for a system?}
\]

First law of thermodynamics: The heat \( Q_H \) added to a system plus the work \( W \) done on the system equals the change in total energy \( E \) of the system.

\[
Q_{net}^{in} + W_{net}^{in} = E_2 - E_1
\]

\[
W_{net}^{in} = W_{pr} + W_{shaft}
\]

\[
\frac{DE}{Dt} = \dot{Q}_{net}^{in} + \dot{W}_{shaft} - \int_{cs} p V \cdot \hat{n} dA
\]
**dE/dt for our System?**

\[
\frac{DE}{Dt} = \dot{Q}_{\text{net}}^{\text{in}} + \dot{W}_{\text{shaft}} - \int p V \cdot \hat{n} dA
\]

**Heat transfer**

\[
\frac{DE}{Dt} = \dot{Q}_{\text{net}}^{\text{in}}
\]

**Pressure work**

\[
\frac{DE}{Dt} = -\int_{cs} p V \cdot \hat{n} dA
\]

**Shaft work**

\[
\frac{DE}{Dt} = \dot{W}_{\text{shaft}}
\]

- \( p = \gamma h \)
- \( F = pA \)
- \( \dot{W}_{\text{pr}} = -F V \)
- \( \dot{W}_{\text{pr}} = -p V A \)
General Energy Equation

\[ \frac{D E}{D t} = \dot{Q}_{\text{net}} + \dot{W}_{\text{shaft}} - \int_{cs} p V \cdot \mathbf{n} \, dA = \frac{\partial}{\partial t} \int_{cv} \rho e \, dV + \int_{cs} \rho e V \cdot \mathbf{n} \, dA \]

\[ \dot{Q}_{\text{net}} + \dot{W}_{\text{shaft}} = \frac{\partial}{\partial t} \int_{cv} e \rho \, d\mathbf{A} + \int_{cs} \left( \frac{p}{\rho} + e \right) \rho V \cdot \mathbf{n} \, dA \]

Total Potential Kinetic Internal (molecular spacing and forces)
Simplify the Energy Equation

\[ \dot{Q}_{\text{net in}} + W_{\text{shaft}} = \frac{\partial}{\partial t} \int e \rho \, dA + \int \left( \frac{p}{\rho} + e \right) \rho \mathbf{V} \cdot \mathbf{n} \, dA \]

Assume...

\[ \frac{p}{\rho} + gz = c \]

\[ e = gz + \frac{V^2}{2} + \tilde{u} \]

\[ \left( q_{\text{net in}} + w_{\text{shaft}} \right) \dot{m} = \int_{cs} \left( \frac{p}{\rho} + gz + \frac{V^2}{2} + \tilde{u} \right) \rho \mathbf{V} \cdot \mathbf{n} \, dA \]

\[ \text{Hydrostatic pressure distribution at cs} \]

\[ \tilde{u} \text{ is uniform over cs} \]

But V is often not uniform over control surface!
Energy Equation: Kinetic Energy

\[ \int_{cs} \left( \frac{V^2}{2} \right) \rho V \cdot \hat{n} \, dA = \alpha \frac{\rho \bar{V}^3 A}{2} \]

*\(V = \) point velocity

*\(\bar{V} = \) average velocity over \(cs\)

If \(V\) tangent to \(n\)

\[ \alpha = \frac{1}{A} \int_{cs} \left( \frac{V^3}{\bar{V}^3} \right) \, dA \]

*\(\alpha = \) kinetic energy correction term

\(\alpha = 1\) for uniform velocity
Energy Equation: steady, one-dimensional, constant density

\[
\left( q_{\text{net in}} + w_{\text{shaft}} \right) \dot{m} = \int_{cS} \left( \frac{p}{\rho} + g z + \frac{V^2}{2} + \tilde{u} \right) \rho \mathbf{V} \cdot \hat{n} \, dA
\]

\[
\int_{cS} \rho \mathbf{V} \cdot \hat{n} \, dA = \dot{m} \quad \text{mass flux rate}
\]

\[
\left( q_{\text{net out}} + w_{\text{shaft}} \right) \dot{m} = \left[ \left( \frac{p_{\text{out}}}{\rho} + g z_{\text{out}} + \alpha \frac{V_{\text{out}}^2}{2} + \tilde{u}_{\text{out}} \right) - \left( \frac{p_{\text{in}}}{\rho} + g z_{\text{in}} + \alpha \frac{V_{\text{in}}^2}{2} + \tilde{u}_{\text{in}} \right) \right] \dot{m}
\]

\[
\frac{p_{\text{in}}}{\rho} + g z_{\text{in}} + \alpha_{\text{in}} \frac{V_{\text{in}}^2}{2} + \tilde{u}_{\text{in}} + q_{\text{net in}} + w_{\text{shaft}} = \frac{p_{\text{out}}}{\rho} + g z_{\text{out}} + \alpha_{\text{out}} \frac{V_{\text{out}}^2}{2} + \tilde{u}_{\text{out}}
\]
Energy Equation: steady, one-dimensional, constant density

\[ \frac{p_{in}}{\rho} + g z_{in} + \alpha_{in} \frac{V_{in}^2}{2} + \tilde{u}_{in} + q_{net in} + w_{shaft} = \frac{p_{out}}{\rho} + g z_{out} + \alpha_{out} \frac{V_{out}^2}{2} + \tilde{u}_{out} \]

divide by g

\[ \frac{p_{in}}{\gamma} + z_{in} + \alpha_{in} \frac{V_{in}^2}{2 g} + \frac{w_{shaft}}{g} = \frac{p_{out}}{\gamma} + z_{out} + \alpha_{out} \frac{V_{out}^2}{2 g} + \frac{\tilde{u}_{out} - \tilde{u}_{in} - q_{net in}}{g} \]

mechanical

thermal

\[ \frac{w_{shaft}}{g} = [h_P] - h_T \]

\[ \frac{\tilde{u}_{out} - \tilde{u}_{in} - q_{net in}}{g} = h_L \]

Lost mechanical energy

\[ \frac{p_{in}}{\gamma} + z_{in} + \alpha_{in} \frac{V_{in}^2}{2 g} + h_P = \frac{p_{out}}{\gamma} + z_{out} + \alpha_{out} \frac{V_{out}^2}{2 g} + h_T + h_L \]
Thermal Components of the Energy Equation

\[ e = gz + \frac{V^2}{2} + \ddot{u} \]

\[ \ddot{u} = c_v T \approx c_p T \]

For incompressible liquids

\[ \ddot{u}_{out} - \ddot{u}_{in} - q_{net}^{in} = h_L \]

Water specific heat = 4184 J/(kg*K)

Change in temperature

\[ \frac{c_p (T_{out} - T_{in}) - q_{net}^{in}}{g} = h_L \]

Heat transferred to fluid

Example
An irrigation pump lifts 50 L/s of water from a reservoir and discharges it into a farmer’s irrigation channel. The pump supplies a total head of 10 m. How much **mechanical** energy is lost? **What is** $h_L$?

**Why can’t I draw the cs at the end of the pipe?**

\[
\frac{p_{in}}{\gamma} + z_{in} + \alpha_{in} \frac{V_{in}^2}{2g} + h_p = \frac{p_{out}}{\gamma} + z_{out} + \alpha_{out} \frac{V_{out}^2}{2g} + h_T + h_L
\]

\[
h_p = z_{out} + h_L
\]

\[
h_L = h_p - z_{out}
\]

\[
h_L = 10 \text{ m} - 4 \text{ m}
\]
The total pipe length is 50 m and is 20 cm in diameter. The pipe length to the pump is 12 m. What is the pressure in the pipe at the pump outlet? You may assume (for now) that the only losses are frictional losses in the pipeline.

\[ h_p = 10 \text{ m} \]

We need **velocity** in the pipe, \( \alpha \), and **head loss**.
Example: Energy Equation
(pressure at pump outlet)

- How do we get the velocity in the pipe?
  \[ Q = VA \quad A = \pi d^2/4 \quad V = 4Q/(\pi d^2) \]
  \[ V = 4(0.05 \text{ m}^3/\text{s})/\left[ \pi (0.2 \text{ m})^2 \right] = 1.6 \text{ m/s} \]

- How do we get the frictional losses?
  Expect losses to be proportional to length of the pipe:
  \[ h_l = (6 \text{ m})(12 \text{ m})/(50 \text{ m}) = 1.44 \text{ m} \]

- What about \( \alpha \)?
Kinetic Energy Correction Term:

\[ \alpha = \frac{1}{A} \int_{cs} \left( \frac{V^3}{V'^3} \right) dA \]

- \( \alpha \) is a function of the velocity distribution in the pipe.
- For a uniform velocity distribution \( \alpha \) is 1
- For laminar flow \( \alpha \) is 2
- For turbulent flow \( 1.01 < \alpha < 1.10 \)
  - Often neglected in calculations because it is so close to 1
Example: Energy Equation

(pressure at pump outlet)

\[ V = 1.6 \text{ m/s} \]
\[ \alpha = 1.05 \]
\[ h_L = 1.44 \text{ m} \]

\[ h_P = \frac{p_{out}}{\gamma} + z_{out} + \alpha_{out} \frac{V_{out}^2}{2g} + h_L \]

\[ p_{out} = \gamma \left( h_P - z_{out} - \alpha_{out} \frac{V_{out}^2}{2g} - h_L \right) \]

\[ p_2 = \left(9810 \text{N/m}^3\right) \left[(10 \text{m}) - (2.4 \text{m}) - (1.05) \frac{(1.6 \text{m/s})^2}{2(9.81 \text{m/s}^2)} - (1.44 \text{m})\right] = 59.1 \text{ kPa} \]
Example: Energy Equation
( Hydraulic Grade Line - HGL )

- We would like to know if there are any places in the pipeline where the pressure is too high ( **pipe burst** ) or too low ( water might boil - cavitation ).

- Plot the pressure as piezometric head ( height water would rise to in a piezometer )

- How?
Example: Energy Equation (Energy Grade Line - EGL)

What is the pressure at the pump intake?

\[
\frac{p_{in}}{\gamma} + z_{in} + \alpha_{in} \frac{V_{in}^2}{2g} + h_P = \frac{p_{out}}{\gamma} + z_{out} + \alpha_{out} \frac{V_{out}^2}{2g} + h_T + h_L
\]

Entrance loss

Exit loss

Loss due to shear

Datum

\( p = 59 \text{ kPa} \)

\( H_P = 10 \text{ m} \)

\( 50 \text{ L/s} \)
EGL (or TEL) and HGL

\[
\text{EGL} = \frac{p}{\gamma} + z + \alpha \frac{V^2}{2g}
\]

\[
\text{HGL} = \frac{p}{\gamma} + z
\]

What is the difference between EGL defined by Bernoulli and EGL defined here?
**EGL (or TEL) and HGL**

- The energy grade line may never be horizontal or slope upward (in direction of flow) unless energy is added (pump).
- The decrease in total energy represents the head loss or energy dissipation per unit weight.
- EGL and HGL are **coincident** and lie at the free surface for water at rest (reservoir).
- Whenever the HGL falls below the point in the system for which it is plotted, the local pressures are lower than the **reference pressure**.
Example HGL and EGL

\[ \frac{p_{in}}{\gamma} + z_{in} + \alpha_{in} \frac{V_{in}^2}{2g} + h_p = \frac{p_{out}}{\gamma} + z_{out} + \alpha_{out} \frac{V_{out}^2}{2g} + h_T + h_L \]
Bernoulli vs. Control Volume

Conservation of Energy

Find the velocity and flow. How would you solve these two problems?

pipe

Free jet
Bernoulli vs. Control Volume

Conservation of Energy

\[
\frac{p_1}{\gamma} + z_1 + \frac{v_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{v_2^2}{2g}
\]

Point to point along streamline
No frictional losses
Based on point velocity

\[
\frac{p_{\text{in}}}{\gamma} + z_{\text{in}} + \alpha_{\text{in}} \frac{V_{\text{in}}^2}{2g} + h_p = \frac{p_{\text{out}}}{\gamma} + z_{\text{out}} + \alpha_{\text{out}} \frac{V_{\text{out}}^2}{2g} + h_T + h_L
\]

Control surface to control surface
Has a term for frictional losses
Based on average velocity
Requires kinetic energy correction factor
Includes shaft work
Has direction!
Power and Efficiencies

- **Electrical power**
  \[ P_{\text{electric}} = EI \]

- **Shaft power**
  \[ P_{\text{shaft}} = T\omega \]

- **Impeller power**
  \[ P_{\text{impeller}} = T\omega \]

- **Fluid power**
  \[ P_{\text{water}} = \gamma QH_p \]

Motor losses

Bearing losses

Pump losses

Prove this!
Example: Hydroplant

Water power = 2.45 MW
Overall efficiency = 0.857
efficiency of turbine = 0.893
efficiency of generator = 0.96

2.45 MW
2100 kW
116 kN·m
50 m
50 m

Q = 5 m³/s
180 rpm
116 kN·m
2100 kW

solution
\[ \frac{p_{in}}{\gamma} + z_{in} + \alpha_{in} \frac{V_{in}^2}{2g} + h_p = \frac{p_{out}}{\gamma} + z_{out} + \alpha_{out} \frac{V_{out}^2}{2g} + h_T + h_L \]

Energy Equation Review

- Control Volume equation
- Simplifications
  - steady
  - constant density
  - hydrostatic pressure distribution across control surface (streamlines parallel)
- Direction of flow matters (in vs. out)
- We don’t know how to predict head loss
Conservation of Energy, Momentum, and Mass

- Most problems in fluids require the use of more than one conservation law to obtain a solution.

- Often a simplifying assumption is required to obtain a solution to heat

- Neglect energy losses (to heat) over a short distance with no flow expansion.

- Neglect shear forces on the solid surface over a short distance.
Head Loss: Minor Losses

- Head (or energy) loss due to:
  - outlets, inlets, bends, elbows, valves, pipe size changes
- Losses due to expansions are **greater** than losses due to contractions. **When \( V \downarrow \), KE \( \rightarrow \) thermal**
- Losses can be minimized by gradual transitions
- Losses are expressed in the form
  \[
  h_L = K_L \frac{V^2}{2g}
  \]
  where \( K_L \) is the loss coefficient
Head Loss due to Sudden Expansion:
Conservation of Energy

Where is $p$ measured? At centroid of control surface

\[
\frac{p_{in}}{\gamma} + z_{in} + \alpha_{in} \frac{V_{in}^2}{2g} + h_P = \frac{p_{out}}{\gamma} + z_{out} + \alpha_{out} \frac{V_{out}^2}{2g} + h_T + h_L
\]

\[
\frac{p_{in} - p_{out}}{\gamma} = \frac{V_{out}^2 - V_{in}^2}{2g} + h_L
\]

\[
h_L = \frac{p_{in} - p_{out}}{\gamma} + \frac{V_{in}^2 - V_{out}^2}{2g}
\]

Relate $V_{in}$ and $V_{out}$? Mass

Relate $p_{in}$ and $p_{out}$? Momentum
Head Loss due to Sudden Expansion: Conservation of Momentum

Apply in direction of flow

Neglect surface shear

Pressure is applied over all of section 1.
Momentum is transferred over area corresponding to upstream pipe diameter.

\[ \rho V_{in}^2 A_{in} \]

\[ \rho V_{out}^2 A_{out} \]

\[ \frac{p_{in} A_{out}^2}{p_{out} A_{out}^2} = \frac{V_{out}^2 - V_{in}^2}{g} \]

\[ \gamma \]
Head Loss due to Sudden Expansion

Energy: \[ h_L = \frac{p_{in} - p_{out}}{\gamma} + \frac{V_{in}^2 - V_{out}^2}{2g} \]

Momentum: \[ p_{in} - p_{out} = \frac{V_{out}^2 - V_{in}^2 \frac{A_{in}}{A_{out}}}{g} \]

Mass: \[ \frac{A_{in}}{A_{out}} = \frac{V_{out}}{V_{in}} \]

\[ h_L = \frac{2V_{out}^2 - 2V_{in}^2 \frac{V_{out}}{V_{in}}}{2g} + \frac{V_{in}^2 - V_{out}^2}{2g} \]

Discharge into a reservoir? \[ K_L = 1 \]
Example: Losses due to Sudden Expansion in a Pipe (Teams!)

A flow expansion discharges 0.5 L/s directly into the air. Calculate the pressure immediately upstream from the expansion.

We can solve this using either the momentum equation or the energy equation (with the appropriate term for the energy losses)!

Use the momentum equation...

\[ V_1 = \frac{0.0005m^3/s}{(0.01m)^2} = 6.4m/s \]

\[ V_2 = 0.71m/s \]

Solution
A scoop attached to a locomotive is used to lift water from a stationary water tank next to the train tracks into a water tank on the train. The scoop pipe is 10 cm in diameter and elevates the water 3 m.

Draw several streamlines in the left half of the stationary water tank (use the scoop as your frame of reference) including the streamlines that attach to the submerged horizontal section of the scoop.

Use the streamlines to help you draw a control volume and clearly label the control surfaces.

How fast must the locomotive be moving ($V_{\text{scoop}}$) to get a flow of 4 L/s if the frictional losses in the pipe are equal to $1.8 V^2/2g$ where $V$ is the average velocity of the water in the pipe. ($V_{\text{scoop}} = 7.7$ m/s)
Scoop

\[ Q = 4 \text{ L/s} \]
\[ d = 10 \text{ cm} \]
\[ V_{\text{scoop}} \]

stationary water tank
Scoop Problem: ‘The Real Scoop’

\[
\frac{p_{in}}{\gamma} + z_{in} + \alpha_{in} \frac{V_{in}^2}{2g} + h_p = \frac{p_{out}}{\gamma} + z_{out} + \alpha_{out} \frac{V_{out}^2}{2g} + h_T + h_L
\]

\[
\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g}
\]

Bernoulli

Energy

moving water tank
Control volumes should be drawn so that the surfaces are either tangent (no flow) or normal (flow) to streamlines.

In order to solve a problem the flow surfaces need to be at locations where all but 1 or 2 of the energy terms are known.

When possible choose a frame of reference so the flows are steady.
Summary

- Control volume equation: Required to make the switch from Lagrangian to Eulerian
- Any conservative property can be evaluated using the control volume equation
  - mass, energy, momentum, concentrations of species
- Many problems require the use of several conservation laws to obtain a solution
Scoop Problem

stationary water tank
Scoop Problem:
Change your Perspective

moving water tank
Scoop Problem: 
Be an Extremist!

Very long riser tube

Very short riser tube
Example: Conservation of Mass
(Team Work)

- The flow through the orifice is a function of the depth of water in the reservoir
  \[ Q = C A_{or} \sqrt{2gh} \]

- Find the time for the reservoir level to drop from 10 cm to 5 cm. The reservoir surface is 15 cm x 15 cm. The orifice is 2 mm in diameter and is 2 cm off the bottom of the reservoir. The orifice coefficient is 0.6.

- CV with constant or changing mass.

- Draw CV, label CS, solve using variables starting with \[ \int_{cs} \rho \mathbf{V} \cdot \hat{n} \, dA = - \frac{\partial}{\partial t} \int_{cv} \rho \, dV \] to integration step
Example Conservation of Mass
Constant Volume

\[
\int_{cs} \rho V \cdot \hat{n} \, dA = -\frac{\partial}{\partial t} \int_{cv} \rho \, dV
\]

\[
\int_{cs_1} \rho_1 V_1 \cdot \hat{n}_1 \, dA + \int_{cs_2} \rho_2 V_2 \cdot \hat{n}_2 \, dA = 0
\]

\[
-V_{res} \, A_{res} + V_{or} \, A_{or} = 0 \quad \text{\(V_{or} \, A_{or} = Q_{or}\)}
\]

\[
V_{res} = -\frac{dh}{dt}
\]

\[
\frac{dh}{dt} \, A_{res} + CA_{or} \sqrt{2gh} = 0
\]
Example Conservation of Mass Changing Volume

\[ \int_{cs} \rho \mathbf{V} \cdot \mathbf{n} \, dA = - \frac{\partial}{\partial t} \int_{cv} \rho \, dV \]

\[ V_{or} A_{or} = - \frac{\partial}{\partial t} \int_{cv} dV \]

\[ V_{or} A_{or} = - \frac{dV}{dt} = - A_{res} \frac{dh}{dt} \]

\[ V_{or} A_{or} = Q_{or} \]

\[ \frac{dh}{dt} A_{res} + CA_{or} \sqrt{2gh} = 0 \]
Example Conservation of Mass

\[
- \frac{A_{res}}{CA_{or} \sqrt{2g}} \int_{h_0}^{h} \frac{dh}{\sqrt{h}} = \int_{0}^{t} dt
\]

\[
- \frac{A_{res}}{CA_{or} \sqrt{2g}} 2 \left( h^{1/2} - h_0^{1/2} \right) = t
\]

\[
\frac{-2 \left( 0.15 m \right)^2}{\left( 0.6 \right) \left( \frac{\pi \left( 0.002 m \right)^2}{4} \right) \sqrt{2 \left( 9.8 m / s^2 \right)}} \left( (0.03 m)^{1/2} - (0.08 m)^{1/2} \right) = t
\]

\[
t = 591 \text{ s}
\]
Pump Head

\[
\frac{p_{in}}{\gamma} + z_{in} + \alpha_{in} \frac{V_{in}^2}{2g} + h_P = \\
\frac{p_{out}}{\gamma} + z_{out} + \alpha_{out} \frac{V_{out}^2}{2g} + h_T + h_L
\]
Example: Venturi
Example: Venturi

Find the flow (Q) given the pressure drop between section 1 and 2 and the diameters of the two sections. Draw an appropriate control volume. You may assume the head loss is negligible. Draw the EGL and the HGL.
Example Venturi

\[ \frac{p_{in}}{\gamma} + z_{in} + \alpha_{in} \frac{V_{in}^2}{2g} + h_P = \frac{p_{out}}{\gamma} + z_{out} + \alpha_{out} \frac{V_{out}^2}{2g} + h_T + h_L \]

\[ \frac{p_{in}}{\gamma} - \frac{p_{out}}{\gamma} = \frac{V_{out}^2}{2g} - \frac{V_{in}^2}{2g} \]

\[ \frac{p_{in}}{\gamma} - \frac{p_{out}}{\gamma} = \frac{V_{out}^2}{2g} \left[ 1 - \left( \frac{d_{out}}{d_{in}} \right)^4 \right] \]

\[ V_{out} = \sqrt{\frac{2g (p_{in} - p_{out})}{\gamma \left[ 1 - (d_{out} / d_{in})^4 \right]}} \]

\[ Q = C_v A_{out} \sqrt{\frac{2g (p_{in} - p_{out})}{\gamma \left[ 1 - (d_{out} / d_{in})^4 \right]}} \]

\[ Q = VA \]

\[ V_{in} A_{in} = V_{out} A_{out} \]

\[ V_{in} \frac{\pi d_{in}^2}{4} = V_{out} \frac{\pi d_{out}^2}{4} \]

\[ V_{in} d_{in}^2 = V_{out} d_{out}^2 \]

\[ V_{in} = V_{out} \frac{d_{out}^2}{d_{in}^2} \]
Reflections

- What is the name of the equation that we used to move from a system (Lagrangian) view to the control volume (Eulerian) view?
- Explain the analogy to your checking account.
- The velocities in the linear momentum equation are relative to …?
- When is “ma” non-zero for a fixed control volume?
- Under what conditions could you generate power from a rotating sprinkler?
- What questions do you have about application of the linear momentum and momentum of momentum equations?
Temperature Rise over Taughanock Falls

- Drop of 50 meters
- Find the temperature rise
- Ignore kinetic energy

\[
c_p (T_{out} - T_{in}) - q_{net}^{in} = h_L
\]

\[
\Delta T = \frac{gh_L + q_{net}^{in}}{c_p}
\]

\[
\Delta T = \frac{(9.8 \text{ m/s}^2)(50 \text{ m})}{4184 \text{ J/Kg} \cdot \text{K}}
\]

\[
\Delta T = 0.117 \text{ K}
\]
Hydropower

\[ P = \gamma Q H_p \]

\[ P_{\text{water}} = \left(9806 \text{ N} / \text{m}^3\right)\left(5 \text{ m}^3 / \text{s}\right)\left(50 \text{ m}\right) = 2.45 \text{ MW} \]

\[ e_{\text{total}} = \frac{2.100 \text{ MW}}{2.45 \text{ MW}} = 0.857 \]

\[ P_{\text{turbine}} = \left(0.116 \text{ MNm}\right)\left(180 \frac{\text{rev}}{\text{min}} \frac{2\pi \text{ rad}}{\text{rev}} \frac{1 \text{ min}}{60 \text{s}}\right) = 2.187 \text{ MW} \]

\[ e_{\text{turbine}} = \frac{2.187 \text{ MW}}{2.45 \text{ MW}} = 0.893 \]

\[ e_{\text{generator}} = \frac{2.100 \text{ MW}}{2.187 \text{ MW}} = 0.96 \]
A flow expansion discharges 0.5 L/s directly into the air. Calculate the pressure immediately upstream from the expansion

\[
p_1 - p_2 = \frac{V_2^2 - V_1^2}{A_1} \frac{A_1}{A_2} \gamma g
\]

\[
p_1 = \frac{V_2^2 - V_1 V_2}{\gamma g}
\]

\[
p_1 = \rho \left( V_2^2 - V_1 V_2 \right)
\]

\[
p_1 = \left( 1000 \text{ kg/s} \right) \left( 0.71 \text{ m/s} \right)^2 - \left( 6.4 \text{ m/s} \right) \left( 0.71 \text{ m/s} \right)
\]

\[
p_1 = -4 \text{ kPa}
\]

\[
V_1 = \frac{0.0005 \text{ m}^3 / \text{s}}{\pi \left( 0.01 \text{ m} \right)^2} = 6.4 \text{ m/s}
\]

\[
V_2 = 0.71 \text{ m/s}
\]

Carburetors and water powered vacuums