Spatial Enhancement

Region operations: \( k'(x, y) = F( k(x-m, y-n), \ldots k(x,y), \ldots k(x+m,y+n) ) \)

Template (Windowing) Operations

Template (window, box, kernel)
- defines the immediate area of interest
- used in spatial domain operations
- usually square/rectangular

Examples of Template Operations
- correlation, convolution
  - spatial smoothing
  - spatial differentiation
  - edge enhancements
- median filter
- variance/ std. deviation
- local area enhancements
- difference operations
- template matching
- logical averaging

Fourier Transform
- Image \( \mapsto \) spatial frequency map (FT)
- frequency domain filtering
- filtered FT map \( \mapsto \) image

Correlation
The correlation of two continuous functions \( f(x) \) and \( g(x) \), denoted by \( f(x) \circ g(x) \), is defined (in 1-dimension) by the relation:

\[
f(x) \circ g(x) = \int_{-\infty}^{+\infty} f(\alpha)g(x + \alpha)d\alpha
\]

Where \( \alpha \) is a dummy variable of integration.

Convolution
The convolution of two continuous functions \( f(x) \) and \( g(x) \), denoted by \( f(x) * g(x) \), is defined (in 1-dimension) by the relation:

\[
f(x) * g(x) = \int_{-\infty}^{+\infty} f(\alpha)g(x - \alpha)d\alpha
\]

Where \( \alpha \) is a dummy variable of integration.
Correlation: graphic description

STEP 1:

Convolution: graphic description

STEP 2:
Convolution & correlation filters

1. **Sum of the filter elements:**
   \[ \sum f(k) = 1 \implies \text{tonal character of the original image is unchanged.} \]
   \[ \sum f(k) > 1 \implies \text{tonal (contrast) stretch} \]
   \[ 0 < \sum f(k) < 1 \implies \text{tonal (contrast) reduction} \]
   \[ \sum f(k) = 0 \implies \text{complete loss of tonal properties.} \]

2. **Laws:** Let \( f_1 \) and \( f_2 \) be filters, and let \( F \) be an image:

   **Distributive Law:** (addition)
   \[
   (f_1 * F) + (f_2 * F) = (f_1 + f_2) * F
   \]
   \[
   (f_1 \circ F) + (f_2 \circ F) = (f_1 + f_2) \circ F
   \]

   **Commutative Law:** (addition)
   \[
   (f_1 + f_2) * F = (f_2 + f_1) * F
   \]
   \[
   (f_1 + f_2) \circ F = (f_2 + f_1) \circ F
   \]

   **Associative Law:** (convolution)
   \[
   f_1 * (f_2 * F) = (f_1 * f_2) * F
   \]

   **Commutative Law:** (convolution)
   \[
   f_1 * f_2 = (f_2 * f_1)
   \]

**Convolution: discrete definition**
The discrete definition of the convolution of a filter \( f(x) \) with image \( g(x) \), is defined as:

\[
(f(i, j) * g(i, j)) = \sum_{k=-(m-1)/2}^{(m-1)/2} \sum_{\ell=-(n-1)/2}^{(n-1)/2} f(k, \ell) g(i-k, j-\ell)
\]

Where:
- \( (i,j) \) = location in the image
- \( (m,n) \) = size of the filter template
- \( (k,\ell) \) = location within the filter template
- \( (k,\ell) = (0,0) \) refers to the filter center

- Filters are usually defined with odd dimensions, i.e., 3x3, 5x5, \ldots
- That way the **center** pixel is always well defined.
- The center (or reference) pixel in a filter with even dimensions is often taken to be the upper left pixel of the center group.

**Spatial Smoothing:**

- **low-pass (averaging or mean) filter** - convolution/correlation operation
  - replaces the image value with a weighted average of the local values

**Linear Stretch:** Convolution/correlation operation using a template with an effective size of 1x1.
- Each image pixel gray value is multiplied by the value of the template.
Median Filter:
- Replaces the image value with the local median
- More generally, a rank-order filter. The filter is applied by sorting the values of in the neighborhood defined by the template, selecting the median value and assigning this value to the pixel.
- For example, in a 3x3 neighborhood, the median is the 5th largest value, in a 5x5 neighborhood the 13th largest value, and so on.

Logical averaging
Local area enhancement (statistical differencing)
- gray-value mapping varies over the image.
- produces same contrast over the entire image.
- let \( k_m(i,j) \) and \( \sigma(i,j) \) and be the mean and standard deviation of gray values in some neighborhood of (i,j).

\[
k_o(i, j) = k_m(i, j) + [k(i, j) - k_m] \frac{\sigma_o}{\sigma(i, j)} \text{ where: } k_o = \text{output gray value}
\]

\[
\sigma = \text{desired standard deviation}
\]

\[
k_m = \text{local mean}
\]

Local area enhancement (forcing a local mean)
- allows more user control over the enhancement

\[
k_o(i, j) = \alpha m_o + (1 - \alpha)[k(i, j) - k_m] \frac{\beta \sigma_o}{\sigma_o + \beta \sigma(i, j)}
\]

\[
\text{where: } k_m(i,j) = \text{local mean}
\]

\[
\sigma(i,j) = \text{local standard deviation}
\]

\[
k_o = \text{output gray value}
\]

\[
m_o = \text{desired mean}
\]

Edge enhancements: 1-D Gradient
- In 1-dimension:

Continuous function:
\[
\nabla_x f = \frac{df}{dx}; \quad \nabla_y f = \frac{df}{dy}
\]

Discrete function:
\[
\nabla_x f = f(i) - f(i-1), \quad \nabla_y f = f(j) - f(j-1)
\]

Edge enhancements: 2-D Gradient
- In 2-dimension:

Continuous function:
\[
\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j}
\]

Discrete function:
\[
\nabla f(i, j) = [f(i, j) - f(i-1, j)] \hat{i} + [f(i, j) - f(i, j-1)] \hat{j}
\]

The Gradient is a vector quantity
**Edge enhancements: Gradient – alternates**

The gradient is a vector quantity, i.e., it has a magnitude and a direction.

Magnitude: $|\nabla f| = \sqrt{(\nabla_x f)^2 + (\nabla_y f)^2}$

Absolute Value: $|\nabla_x f| + |\nabla_y f|

Robert's gradient:

$[f(i,j) - f(i+1,j+1)]^2 + [f(i+1,j) - f(i,j+1)]^2$

Sum of the directional derivatives: $\nabla_x f + \nabla_y f$

| -1 1 0 | 0 -1 0 |
| 0 0 0 | 0 1 0 |

| 0 -1 0 | = | -1 2 0 |
| 0 0 0 | 0 0 0 |

**Edge Enhancement: Laplacian**

Continuous function: $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

Discrete function:

$\frac{\partial^2 f}{\partial x^2} = f(i-1,j) - 2f(i,j) + f(i+1,j)$

$\frac{\partial^2 f}{\partial y^2} = f(i,j-1) - 2f(i,j) + f(i,j+1)$

Discrete 2D Laplacian: $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = f(i-1,j) + f(i+1,j) + f(i,j-1) + f(i,j+1) + 4f(i,j)$

| 0 0 0 | 0 1 0 | 0 0 0 |
| 1 -2 1 | 0 -2 0 | 1 -4 1 |
| 0 0 0 | 0 1 0 | 0 0 0 |
**Convolution filter characteristics:**

- **Sum of the filter elements:**
  
  \[ \sum f(k) = 1 \implies \text{tonal character of the original image is unchanged.} \]
  
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  \[ \sum f(k) = 0 \implies \text{complete loss of tonal properties.} \]

- **Directionality/symmetry**

![Diagram showing the convolution filter characteristics with various sum and directionality/symmetry results.](image-url)