P4.19 An incompressible flow field has the cylindrical velocity components $v_\theta = Cr$, $v_z = K(R^2 - r^2)$, $v_r = 0$, where $C$ and $K$ are constants and $r \leq R$, $z \leq L$. Does this flow satisfy continuity? What might it represent physically?
Solution: We check the incompressible continuity relation in cylindrical coordinates:

$$\frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 = 0 + 0 + 0 \quad \text{satisfied identically} \quad \text{Ans.}$$

This flow also satisfies (cylindrical) momentum and could represent laminar flow inside a tube of radius $R$ whose outer wall ($r = R$) is rotating at uniform angular velocity.
P4.20 A two-dimensional incompressible velocity field has \( u = K(1 - e^{-\alpha y}) \), for \( x \leq L \) and \( 0 \leq y \leq \infty \). What is the most general form of \( v(x, y) \) for which continuity is satisfied and \( v = v_0 \) at \( y = 0 \)? What are the proper dimensions for constants \( K \) and \( \alpha \)?

Solution: We can find the appropriate velocity \( v \) from two-dimensional continuity:

\[
\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{\partial}{\partial x} [K(1 - e^{-\alpha y})] = 0, \quad \text{or:} \quad v = \text{fcn}(x) \text{ only}
\]

Since \( v = v_0 \) at \( y = 0 \) for all \( x \), then it must be that \( v = v_0 = \text{const} \quad \text{Ans.} \)

The dimensions of \( K \) are \( \{K\} = \{L/T\} \) and the dimensions of \( \alpha \) are \( \{L^{-1}\} \). \quad \text{Ans.}
P4.27 A frictionless, incompressible steady-flow field is given by

\[ \mathbf{V} = 2xy\mathbf{i} - y^2\mathbf{j} \]

in arbitrary units. Let the density be \( \rho_0 = \) constant and neglect gravity. Find an expression for the pressure gradient in the \( x \) direction.

Solution: For this (gravity-free) velocity, the momentum equation is

\[ \rho \left( u \frac{\partial \mathbf{V}}{\partial x} + v \frac{\partial \mathbf{V}}{\partial y} \right) = -\nabla p, \quad \text{or} \quad \rho_0 [(2xy)(2y)\mathbf{i} + (-y^2)(2x\mathbf{i} - 2y\mathbf{j})] = -\nabla p \]

Solve for \( \nabla p = -\rho_0 (2xy^2\mathbf{i} + 2y^3\mathbf{j}), \quad \text{or} \quad \frac{\partial p}{\partial x} = -\rho_0 2xy^2 \quad \text{Ans.} \)
P4.28 Consider the incompressible flow field of Prob. P4.6, with velocity components \( u = 2y, v = 8x, w = 0 \). Neglect gravity and assume constant viscosity. (a) Determine whether this flow satisfies the Navier-Stokes equations. (b) If so, find the pressure distribution \( p(x, y) \) if the pressure at the origin is \( p_0 \).

Solution: In Prob. P4.6 we found the accelerations, so we can proceed to Navier-Stokes:

\[
\rho (u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}) = \rho [0 + (8x)(2)] = -\frac{\partial p}{\partial x} + \rho g_x + \mu \nabla^2 u = -\frac{\partial p}{\partial x} + 0 + 0; \quad \frac{\partial p}{\partial x} = -16 \rho x
\]

\[
\rho (u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y}) = \rho [(2y)(8) + 0] = -\frac{\partial p}{\partial y} + \rho g_y + \mu \nabla^2 v = -\frac{\partial p}{\partial y} + 0 + 0; \quad \frac{\partial p}{\partial y} = -16 \rho y
\]

Noting that \( \frac{\partial^2 p}{\partial x \partial y} = 0 \) in both cases, we conclude **Yes, satisfies Navier-Stokes.** Ans.(a)

(b) The pressure gradients are simple, so we may easily integrate:

\[
dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy, \text{ or: } p = \int -16 \rho x dx + \int -16 \rho y dy = -8 \rho (x^2 + y^2) + \text{const}
\]

If \( p(0,0) = p_0 \), then \( p = p_0 - 8 \rho (x^2 + y^2) \) Ans.(b)

This is an exact solution, but it is not Bernoulli’s equation. The flow is *rotational*. 
P4.80 An oil film drains steadily down the side of a vertical wall, as shown. After an initial development at the top of the wall, the film becomes independent of \( z \) and of constant thickness. Assume that \( w = w(x) \) only that the atmosphere offers no shear resistance to the film. (a) Solve Navier-Stokes

for \( w(x) \). (b) Suppose that film thickness and \( \left[ \frac{\partial w}{\partial x} \right] \) at the wall are measured. Find an expression which relates \( \mu \) to this slope \( \left[ \frac{\partial w}{\partial x} \right] \).

**Solution:** First, there is no pressure gradient \( \frac{\partial p}{\partial z} \) because of the constant-pressure atmosphere. The Navier-Stokes \( z \)-component is \( \mu \left( \frac{\partial^2 w}{\partial x^2} \right) = \rho g \), and the solution requires \( w = 0 \) at \( x = 0 \) and \( (dw/dx) = 0 \) (no shear at the film edge) at \( x = \delta \). The solution is:

\[
    w = \frac{\rho g x}{2 \mu} (x - 2 \delta) \quad \text{Ans. (a)}
\]

NOTE: \( w \) is negative (down)

The wall slope is \( dw/dx \big|_{wall} = -\rho g \delta / \mu \), rearrange: \( \mu = -\rho g \delta \left[ \frac{dw}{dx} \big|_{wall} \right] \quad \text{Ans. (b)} \)
P4.84 Consider a viscous film of liquid draining uniformly down the side of a vertical rod of radius $a$, as in Fig. P4.84. At some distance down the rod the film will approach a terminal or *fully developed* draining flow of constant outer radius $b$, with $v_z = v_z(r)$, $v_\theta = v_\phi = 0$. Assume that the atmosphere offers no shear resistance to the film motion. Derive a differential equation for $v_z$, state the proper boundary conditions, and solve for the film velocity distribution. How does the film radius $b$ relate to the total film volume flow rate $Q$?

**Solution:** With $v_z = fcn(r)$ only, the Navier-Stokes $z$-momentum relation is

$$\rho \frac{dv_z}{dt} = -\frac{\partial p}{\partial z} + \rho g + \mu \nabla^2 v_z,$$

or:

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{dv_z}{dr} \right) = -\frac{\rho g}{\mu},$$

Integrate twice:

$$v_z = -\frac{\rho g r^2}{4\mu} + C_1 \ln(r) + C_2$$

The proper B.C. are: $u(a) = 0$ (no-slip) and $\mu \frac{\partial v_z}{\partial r}(b) = 0$ (no free-surface shear stress)

The final solution is

$$v_z = \frac{\rho g b^2}{2\mu} \ln \left( \frac{r}{a} \right) - \frac{\rho g}{4\mu} \left( r^2 - a^2 \right) \quad Ans.$$  

The flow rate is

$$Q = \int_a^b v_z 2\pi r \, dr = \frac{\pi \rho g a^4}{8\mu} \left( -3\sigma^4 - 1 + 4\sigma^2 + 4\sigma^4 \ln \sigma \right),$$

where $\sigma = \frac{b}{a} \quad Ans.$