P4.7 Consider a sphere of radius $R$ immersed in a uniform stream $U_0$, as shown in Fig. P4.7. According to the theory of Chap. 8, the fluid velocity along streamline $AB$ is given by

$$V = u i = U_0 \left(1 + \frac{R^3}{x^3}\right) i$$

Find (a) the position of maximum fluid acceleration along $AB$ and (b) the time required for a fluid particle to travel from $A$ to $B$. Note that $x$ is negative along line $AB$.

**Solution:** (a) Along this streamline, the fluid acceleration is one-dimensional:

$$\frac{du}{dt} = u \frac{\partial u}{\partial x} = U_0 (1 + R^3/x^3)(-3U_0R^3/x^4) = -3U_0R^3(x^{-4} + R^3x^{-7}) \quad \text{for } x \leq -R$$

The maximum occurs where $d(ax)/dx = 0$, or at $x = -(7R^3/4)^{1/3} \approx -1.205R$  \textit{Ans.} (a)

(b) The time required to move along this path from $A$ to $B$ is computed from

$$u = \frac{dx}{dt} = U_0 (1 + R^3/x^3), \quad \text{or: } \int_{-R}^{-4R} \frac{dx}{1 + R^3/x^3} = \int_{0}^{1} U_0 \, dt, \quad \text{or: } \quad U_0 t = \left[ x - \frac{R}{6} \ln \frac{(x + R)^2}{x^2 - Rx + R^2} + \frac{R}{\sqrt{3}} \tan^{-1} \left( \frac{2x - R}{R\sqrt{3}} \right) \right]_{-4R}^{-R} = \infty$$

It takes \textit{an infinite time} to actually \textit{reach} the stagnation point, where the velocity is zero. \textit{Ans.} (b)

P4.8 When a valve is opened, fluid flows in the expansion duct of Fig. P4.8 according to the approximation

$$V = iU \left(1 - \frac{x}{2L}\right) \tanh \frac{Ut}{L}$$
Find (a) the fluid acceleration at \((x, t) = (L, L/U)\) and (b) the time for which the fluid acceleration at \(x = L\) is zero. Why does the fluid acceleration become negative after condition (b)?

![Graph](image)

**Solution:** This is a one-dimensional *unsteady* flow. The acceleration is

\[
a_x = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = U \left(1 - \frac{x}{2L}\right) \frac{U}{L} \text{sech}^2 \left(\frac{Ut}{L}\right) - U \left(1 - \frac{x}{2L}\right) \frac{U}{2L} \tanh \left(\frac{Ut}{L}\right)
\]

\[
= \frac{U^2}{L} \left(1 - \frac{x}{2L}\right) \left[ \text{sech}^2 \left(\frac{Ut}{L}\right) - \frac{1}{2} \tanh^2 \left(\frac{Ut}{L}\right) \right]
\]

At \((x, t) = (L, L/U)\), \(a_x = (U^2/L)(1/2)[\text{sech}^2(1) - 0.5\tanh^2(1)] \approx 0.0650 \frac{U^2}{L}\) \text{Ans. (a)}

The acceleration becomes zero when

\[
\text{sech}^2 \left(\frac{Ut}{L}\right) = \frac{1}{2} \tanh^2 \left(\frac{Ut}{L}\right), \quad \text{or } \sinh^2 \left(\frac{Ut}{L}\right) = 2,
\]

\[
\text{or } \frac{Ut}{L} = 1.146 \quad \text{Ans. (b)}
\]

The acceleration starts off positive, then goes through zero and turns negative as the negative *convective* acceleration overtakes the decaying positive *local* acceleration.

**P4.9** An idealized incompressible flow has the proposed three-dimensional velocity distribution

\[
V = 4xy^2i + f(y)j - yz^2k
\]
**Solution:** With no $\theta$ variation and no $v_0$, the equation of continuity (4.9) becomes
\[
\frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{\partial v_z}{\partial z} = 0 = \frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{\partial}{\partial z} (Bz),
\]
or: \(\frac{\partial}{\partial r} (r v_r) = -Br\); Integrate: \(r v_r = -\frac{B}{2} r^2 + f(z)\)

Finally, \(v_r = -\frac{B}{2} r + \frac{f(z)}{r}\) \(\text{Ans.}\)

The “function of integration”, \(f(z)\), is arbitrary, at least until boundary conditions are set.

---

**P4.23** A tank volume $\mathcal{V}$ contains gas at conditions $(\rho_0, p_0, T_0)$. At time $t = 0$ it is punctured by a small hole of area $A$. According to the theory of Chap. 9, the mass flow out of such a hole is approximately proportional to $A$ and to the tank pressure. If the tank temperature is assumed constant and the gas is ideal, find an expression for the variation of density within the tank.

**Solution:** This problem is a realistic approximation of the “blowdown” of a high-pressure tank, where the exit mass flow is choked and thus proportional to tank pressure. For a control volume enclosing the tank and cutting through the exit jet, the mass relation is
\[
\frac{d}{dt} (m_{\text{tank}}) + m_{\text{exit}} = 0, \quad \text{or:} \quad \frac{d}{dt} (\rho v) = -m_{\text{exit}} = -CpA, \quad \text{where} \quad C = \text{constant}
\]

Introduce $\rho = \frac{p}{RT_0}$ and separate variables: \[\int_{p_0}^{p(t)} \frac{dp}{p} = -\frac{C R T_0 A}{\nu} \int_0^t dt\]

The solution is an exponential decay of tank density: \(p = p_0 \exp(-C R T_0 A t / \nu)\). \(\text{Ans.}\)

---

**P4.24** For incompressible laminar flow between parallel plates (see Fig. 4.12b), the flow is two-dimensional ($v \neq 0$) if the walls are porous. A special case solution is $u = (A - Bx)(h^2 - y^2)$, where $A$ and $B$ are constants. \(a\) Find a general formula for velocity $v$ if $v = 0$ at $y = 0$. \(b\) What is the value of the constant $B$ if $v = v_w$ at $y = +h$?

**Solution:** \(a\) Use the equation of continuity to find the velocity $v$:
\[ \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -(-B)(h^2 - y^2) \]

Integrate: \[ v = B \int (h^2 - y^2) \, dy = B(h^2y - \frac{y^3}{3}) + f(x) \]

If \( v=0 \) at \( y=0 \), then \( f(x) = 0 \). \( \therefore \) \( v = B(h^2y - \frac{y^3}{3}) \) \( \text{Ans.}(a) \)

(b) Just simply introduce this boundary condition into the answer to part (a):

\[ v(y = h) = v_w = B(h^2 - \frac{h^3}{3}) \text{, hence } B = \frac{3v_w}{2h^3} \text{ Ans.}(b) \]

\[ \text{P4.25} \quad \text{An incompressible flow in polar coordinates is given by} \]

\[ v_r = K \cos \theta \left( 1 - \frac{b}{r^2} \right) \]

\[ v_\theta = -K \sin \theta \left( 1 + \frac{b}{r^2} \right) \]

Does this field satisfy continuity? For consistency, what should the dimensions of constants \( K \) and \( b \) be? Sketch the surface where \( v_r = 0 \) and interpret.

\[ \text{Solution: } \text{Substitute into plane polar coordinate continuity:} \]

\[ \frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} = 0 = \frac{1}{r} \frac{\partial}{\partial r} \left[ K \cos \theta \left( r - \frac{b}{r} \right) \right] + \frac{1}{r} \frac{\partial}{\partial \theta} \left[ -K \sin \theta \left( 1 + \frac{b}{r^2} \right) \right] = 0 \text{ Satisfied} \]

The dimensions of \( K \) must be velocity, \( \{K\} = \{L/T\} \), and \( b \) must be area, \( \{b\} = \{L^2\} \). The surfaces where \( v_r = 0 \) are the y-axis and the circle \( r = \sqrt{b} \), as shown above. The pattern represents inviscid flow of a uniform stream past a circular cylinder (Chap. 8).
Solution for problem 6).

a). According to the continuity equation (two-d flow, \( \frac{\partial \rho}{\partial y} = 0 \))
\[
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0
\]
we have
\[
\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} = -\frac{\partial}{\partial x} \left[ U_0 \left( \frac{2z}{C\sqrt{x}} - \frac{z^2}{C^2x} \right) \right]
= -\frac{2U_0 z}{C} \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{x}} \right) + \frac{U_0 z^2}{C^2} \frac{\partial}{\partial x} \left( \frac{1}{x} \right)
= \frac{U_0 z}{C} x^{-\frac{3}{2}} - \frac{U_0 z^2}{C^2} x^{-2}
\]
Solving the above partial differential equation we have
\[
w(x, z) = \frac{U_0 x^{-\frac{3}{2}} z^2}{2C} - \frac{U_0 x^{-2} z^3}{3C^2} + C_1(x)
\]
Due to the no-slip boundary condition, \( w(x, z = 0) = 0 \). Substitute it into the above general solution we have
\[
w(x, z) = \frac{U_0 x^{-\frac{3}{2}} z^2}{2C} - \frac{U_0 x^{-2} z^3}{3C^2}
\]
At \( x = 1 \) we can define \( w_1(z) = w(x = 1, z) \), such that
\[
w_1(z) = \frac{U_0 z^2}{2C} - \frac{U_0 z^3}{3C^2}
\]
Thus we have
\[
\frac{d w_1}{dz} = \frac{U_0 z}{C} - \frac{U_0 z^2}{C^2}
\]
and
\[
\frac{d^2 w_1}{dz^2} = \frac{U_0}{C} - \frac{2U_0 z}{C^2}
\]
Therefore \( w_1(z) \) have maximum or minimum values at \( z = 0 \) and \( z = C \). Since we have
\[
\frac{d^2 w_1}{dz^2} \bigg|_{z=0} = \frac{U_0}{C} > 0
\]
\[
\frac{d^2 w_1}{dz^2} \bigg|_{z=C} = \frac{U_0}{C} - \frac{2U_0 C}{C^2} = \frac{U_0}{C} < 0
\]
Thus the maximum value of \( w_1(z) \) is \( w_1(z = C) \), hence
\[
w(x = 1, z)_{\text{max}} = w(x = 1, z = C) = \frac{U_0 C}{2} - \frac{U_0 C}{3}
= \frac{U_0 C}{6} = \frac{3 \times 0.011}{6} = 0.0055 \text{ m/s}
\]
\[ p = p(0,0,0) - \rho g z - (\rho / 2)R^2(x^2 + y^2 + 4z^2) \quad \text{Ans. (b)} \]

Note that the last term is identical to \((\rho / 2)(u^2 + v^2 + w^2)\), in other words, Bernoulli’s equation.

(c) For irrotational flow, the curl of the velocity field must be zero:

\[ \nabla \times \mathbf{V} = \mathbf{i}(0 - 0) + \mathbf{j}(0 - 0) + \mathbf{k}(0 - 0) = 0 \quad \text{Yes, irrotational.} \quad \text{Ans. (c)} \]

P4.35 From the Navier-Stokes equations for incompressible flow in polar coordinates (App. E for cylindrical coordinates), find the most general case of purely circulating motion \(v_\theta(r)\), for flow with no slip between two fixed concentric cylinders, as in Fig. P4.35.

Solution: The preliminary work for this problem is identical to Prob. 4.32 on an earlier page. That is, there are two possible solutions for purely circulating motion \(v_\theta(r)\), hence

\[ v_\theta = C_1 r + \frac{C_2}{r}, \quad \text{subject to} \quad v_\theta(a) = 0 = C_1 a + C_2 / a \quad \text{and} \quad v_\theta(b) = 0 = C_1 b + C_2 / b \]

This requires \(C_1 = C_2 = 0\), or \(v_\theta = 0\) (no steady motion possible between fixed walls) \(\text{Ans.}\).

P4.36 A constant-thickness film of viscous liquid flows in laminar motion down a plate inclined at angle \(\theta\), as in Fig. P4.36. The velocity profile is

\[ u = Cy(2h - y) \quad v = w = 0 \]

Find the constant \(C\) in terms of the specific weight and viscosity and the angle \(\theta\). Find the volume flux \(Q\) per unit width in terms of these parameters.
Solution: There is atmospheric pressure all along the surface at \( y = h \), hence \( \frac{\partial p}{\partial x} = 0 \). The x-momentum equation can easily be evaluated from the known velocity profile:

\[
\rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = - \frac{\partial p}{\partial x} + \rho g_s + \mu \nabla^2 u, \quad \text{or:} \quad 0 = 0 + \rho g \sin \theta + \mu (-2C)
\]

Solve for \( C = \frac{\rho g \sin \theta}{2\mu} \) \( \text{Ans. (a)} \)

The flow rate per unit width is found by integrating the velocity profile and using \( C \):

\[
Q = \int_0^h u \, dy = \int_0^h Cy(2h - y) \, dy = \frac{2}{3} Ch^3 = \frac{\rho gh^3 \sin \theta}{3\mu} \quad \text{per unit width} \quad \text{Ans. (b)}
\]

P4.37 A viscous liquid of constant density and viscosity falls due to gravity between two parallel plates a distance \( 2h \) apart, as in the figure. The flow is fully developed, that is, \( w = w(x) \) only. There are no pressure gradients, only gravity. Set up and solve the Navier-Stokes equation for the velocity profile \( w(x) \).

Solution: Only the \( z \)-component of Navier-Stokes is relevant:

\[
\rho \frac{dw}{dt} = 0 = \rho g + \mu \frac{d^2 w}{dx^2}, \quad \text{or:} \quad w'' = - \frac{\rho g}{\mu}, \quad w(-h) = w(+h) = 0 \quad (\text{no-slip})
\]

The solution is very similar to Eqs. (4.142) to (4.143) of the text:

\[
w = \frac{\rho g}{2\mu} (h^2 - x^2) \quad \text{Ans.}
\]

P4.38 Show that the incompressible flow distribution, in cylindrical coordinates,

\[
v_r = 0 \quad v_\theta = Cr^n \quad v_z = 0
\]
Before integrating again, note that \( \frac{dT}{dy} = 0 \) at \( y = h/2 \) (the symmetry condition), so \( C_1 = -h^3/6 \). Now integrate once more:

\[
T = -\frac{16\mu u_{\max}^2}{kh^4} \left( \frac{h^2 y^2}{2} - 2h \frac{y^3}{3} + \frac{y^4}{4} + C_1 y \right) + C_2
\]

If \( T = T_w \) at \( y = 0 \) and at \( y = h \), then \( C_2 = T_w \). The final solution is:

\[
T = T_w + \frac{8\mu u_{\max}^2}{k} \left[ \frac{y}{3h} - \frac{y^2}{h^2} + \frac{4y^3}{3h^3} - \frac{2y^4}{3h^4} \right]
\]

Ans.

This is exactly the same solution as Problem P4.40 above, except that, here, the coordinate \( y \) is measured from the bottom wall rather than the centerline.

**P4.42** Suppose that we wish to analyze the rotating, partly-full cylinder of Fig. 2.23 as a *spin-up* problem, starting from rest and continuing until solid-body-rotation is achieved. What are the appropriate boundary and initial conditions for this problem?

**Solution:** Let \( V = V(r, z, t) \). The initial condition is: \( V(r, z, 0) = 0 \). The boundary conditions are:

Along the side walls: \( v_\theta(R, z, t) = R\Omega, \quad v_r(R, z, t) = 0, \quad v_z(R, z, t) = 0 \).

At the bottom, \( z = 0 \): \( v_\theta(r, 0, t) = r\Omega, \quad v_r(r, 0, t) = 0, \quad v_z(r, 0, t) = 0 \).

At the free surface, \( z = \eta \): \( p = p_{\text{atm}}, \quad \pi z = \tau_{\theta r} = 0 \).

**P4.43** For the draining liquid film of Fig. P4.36, what are the appropriate boundary conditions (a) at the bottom \( y = 0 \) and (b) at the surface \( y = h \)?

**Solution:** The physically realistic conditions at the upper and lower surfaces are:

(a) at the bottom, \( y = 0 \), no-slip: \( u(0) = 0 \) Ans. (a)

(b) At the surface, \( y = h \), no shear stress, \( \mu \frac{\partial u}{\partial y} = 0 \), or \( \frac{\partial u}{\partial y}(h) = 0 \) Ans. (b)
Solution: With \( v_z = fcn(r) \) only, the Navier-Stokes \( z \)-momentum relation is

\[
\rho \frac{dv_z}{dt} = 0 = - \frac{\partial p}{\partial z} + \rho g + \mu \nabla^2 v_z,
\]

or:

\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{dv_z}{dr} \right) = - \frac{\rho g}{\mu}, \quad \text{Integrate twice:} \quad v_z = - \frac{\rho g r^2}{4\mu} + C_1 \ln(r) + C_2
\]

The proper B.C. are: \( u(a) = 0 \) (no-slip) and \( \mu \frac{\partial v_z}{\partial x}(b) = 0 \) (no free-surface shear stress)

The final solution is \( v_z = \frac{\rho g b^2}{2\mu} \ln \left( \frac{r}{a} \right) - \frac{\rho g}{4\mu} (r^2 - a^2) \quad \text{Ans.} \)

The flow rate is \( Q = \int_a^b v_z 2\pi r \, dr = \frac{\pi \rho g a^4}{8\mu} (-3\sigma^4 - 1 + 4\sigma^2 + 4\sigma^4 \ln \sigma), \)

where \( \sigma = \frac{b}{a} \quad \text{Ans.} \)

P4.85 A flat plate of essentially infinite width and breadth oscillates sinusoidally in its own plane beneath a viscous fluid, as in Fig. P4.85. The fluid is at rest far above the plate. Making as many simplifying assumptions as you can, set up the governing differential equation and boundary conditions for finding the velocity field \( u \) in the fluid. Do not solve (if you can solve it immediately, you might be able to get exempted from the balance of this course with credit).

Solution: Assume \( u = u(y, t) \) and \( \partial p / \partial x = 0 \). The \( x \)-momentum relation is

\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = - \frac{\partial p}{\partial x} + \rho g_x + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),
\]

or:

\[
\rho \left( \frac{\partial u}{\partial t} + 0 + 0 \right) = 0 + 0 + \mu \left( 0 + \frac{\partial^2 u}{\partial y^2} \right), \quad \text{or, finally:}
\]

\[
\frac{\partial u}{\partial t} = \frac{\mu}{\rho} \frac{\partial^2 u}{\partial y^2} \quad \text{subject to:} \quad u(0, t) = U_0 \sin(\omega t) \quad \text{and} \quad u(\infty, t) = 0. \quad \text{Ans.}
\]
Rotating a cylinder in a large expanse of fluid sets up (eventually) a potential vortex flow.

**P4.95** Two immiscible liquids of equal thickness $h$ are being sheared between a fixed and a moving plate, as in Fig. P4.95. Gravity is neglected, and there is no variation with $x$.

Find an expression for (a) the velocity at the interface; and (b) the shear stress in each fluid. Assume steady laminar flow.

**Solution:** Treat this as a Ch. 4 problem (not Ch. 1), use continuity and Navier-Stokes:

Continuity: \( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 + \frac{\partial v}{\partial y} = 0 \); thus $v = const = 0$ for no-slip at the walls.

This tells us that there is no velocity $v$, hence we need only consider $u(y)$ in Navier-Stokes:

\[
\rho_{1,2} \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu_{1,2} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
\]

or: $0 + 0 = 0 + \mu_{1,2} (0 + \frac{d^2 u}{dy^2})$

Thus $u = a + by$
The velocity profiles are linear in y but have a different slope in each layer. Let \( u_I \) be the velocity at the interface. (a) The shear stress is the same in each layer:

\[
\tau = \mu_1 \frac{u_I}{h} = \mu_2 \frac{V - u_I}{h} \quad \text{Solve for} \quad u_I = \frac{\mu_2}{\mu_1 + \mu_2} V \quad \text{Ans.}(a)
\]

(b) In terms of the upper plate velocity, \( V \), the shear stress is

\[
\tau = \left( \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} \right) \frac{V}{h} \quad \text{Ans.}(b)
\]

**P4.96** Reconsider Prob. P1.44 and calculate (a) the inner shear stress and (b) the power required, if the exact laminar-flow formula, Eq. (4.140) is used. (c) Determine whether this flow pattern is stable. [HINT: The shear stress in \((r, \theta)\) coordinates is *not* like plane flow.]

**Solution:** The exact laminar-flow velocity is Eq. (4.140), and the shear stress is Eq. (D.9):

\[
v_\theta = \Omega_i r_i \left[ \frac{(r_o / r) - (r / r_o)}{(r_o / r_i) - (r_i / r_o)} \right]
\]

\[
\tau_{r\theta} = \mu \left( \frac{dv_\theta}{dr} - \frac{v_\theta}{r} \right) = \mu \left[ \frac{\Omega_i r_i}{(r_o / r_i) - (r_i / r_o)} \right] \frac{2r_o}{r^2}
\]

Recall the data from Prob. P1.44: \( r_i = 5 \text{ cm}, r_o = 6 \text{ cm}, L = 120 \text{ cm}, \mu = 0.86 \text{ kg/m-s} \) (SAE 50W oil), and \( \Omega_i = 900 \text{ rev/min} = 94.25 \text{ rad/s} \). At the inner cylinder,

\[
\tau_{\text{inner}} = \mu \left[ \frac{\Omega_i r_i}{(r_o / r_i) - (r_i / r_o)} \right] \frac{2r_o}{r_i^2} = (0.86) \left[ \frac{94.25(0.05)}{0.06/0.05 - 0.05/0.06} \right] \frac{2(0.06)}{(0.05)^2} = 531 \text{ Pa} \quad \text{Ans.}(a)
\]

The moment and power required are

\[
M = \tau_i 2\pi r_i^2 L = (531) 2\pi (0.05)^2 (1.20) = 10.0 \text{ N-m}
\]

\[
\text{Power} = \Omega_i M = (94.25 \text{ rad/s}) (10.0 \text{ N-m}) = 943 \text{ watts} \quad \text{Ans.}(b)
\]
Let's work in cylindrical coordinates.

The Navier-Stokes equations in cylindrical coordinates given in Section 6.8.2 in text.

Assume flow in z-direction only, \( \dot{V}_0 = \dot{V}_r = 0 \)

Also assume stems' state, \( \dot{z} = 0 \)

### Full Equations

- **r normally**
  \[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_r}{\partial r} \right) = \frac{\partial p}{\partial r} \]
  \[ \Rightarrow p = \text{const} (r, z) \]

- **\( \theta \) normally**
  \[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_r}{\partial r} \right) = \frac{\partial p}{\partial \theta} \]
  \[ \Rightarrow p = \text{const} (r, z) \]

- **z normally**
  \[ \frac{\partial u_z}{\partial z} = -\frac{2\pi}{\lambda} - \rho g + \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial z^2} \right] \]

But, continuity gives:
\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r u_r \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \]

Also, assume flow is symmetric, \( \frac{\partial v_r}{\partial \theta} = 0 \)

- **\( \theta \) normally**
  \[ 0 = -\frac{\partial u_r}{\partial z} - \rho g + \mu \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_r}{\partial r} \right) \]

\[ \frac{\text{d}^2 u_r}{\text{d}z^2} = 0 \]

### Integrating in \( r \)

\[ \frac{2\pi}{\lambda} + \rho g \left( \frac{r}{r} \right) \frac{\partial u_r}{\partial r} \]

### Integrating in \( r \) again:

\[ \frac{2\pi}{\lambda} \frac{\partial u_r}{\partial r} \]

At \( r = 0 \) \( \dot{V}_r \) is bounded \( \Rightarrow C_1 = 0 \)

At \( r = R \) \( \dot{V}_r = 0 \) (no slip condition) \( \Rightarrow C_2 = \frac{\rho + \rho g}{14} R^2 \)
\[ V_z = \frac{[L + \rho g]}{[L + \rho g]} \left( \frac{r^2 - R^2}{4} \right) \]

\[ Q = \sum_{A} v_z \cdot dA = -\int_{0}^{R} v_z \cdot 2\pi r \, dr \]

\[ Q = \int_{0}^{R} \left[ \frac{L + \rho g}{4} \right] \left( \frac{r^2 - R^2}{4} \right) 2\pi r \, dr \]

\[ = \left[ \frac{L + \rho g}{4} \right] \frac{1}{2} \left( \frac{R^4}{2} - \frac{R^4}{4} \right) \left[ \frac{L + \rho g}{4} \right] \frac{1}{2} \left( \frac{R^4}{2} - \frac{R^4}{4} \right) \]

\[ Q = \frac{[L + \rho g]}{8} \frac{R^4}{4} \]

\[ \therefore Q = \frac{[L + \rho g]}{8} \frac{R^4}{4} \]

a) From result for \( Q \), we see that if \( \frac{L}{\rho g} >> 1 \)

\[ \text{then we can ignore } g \Rightarrow \text{yes} \]

Re-arranging, we have:

\[ \frac{L}{\rho g} >> 1 \]

\[ \Rightarrow \frac{L}{\rho g} >> 1 \]

\[ \left( \frac{L}{\rho g} \right)^2 >> 1 \]

\[ \Rightarrow \frac{L}{\rho g} >> 1 \]

\[ \text{Note - a formal dimensional analysis can show this too but must cast } g \text{ nondimensional in terms of } L, \text{ not } V \text{ (which gives the traditional Fresnel number which is of little help here).} \]