1.11 Review

- \( \tau = \mu \frac{du}{dz} \) where in a small gap we assume \( \frac{du}{dz} \) is a constant (e.g., the velocity profile is linear) \( \Rightarrow \) therefore \( \frac{du}{dz} = \frac{\Delta u}{\Delta z} \)

- Vapor pressure: High velocity \( \Rightarrow \) low pressure \( \Rightarrow \) cavitation.

1.12 The Velocity Field - From Two Perspectives

The *Eulerian* velocity is the velocity with respect to a fixed position. E.g., you stand on a hill with a velocity sensor and record the three components of velocity as a function of time. This is the Eulerian velocity vector at that physical point. We write the Eulerian velocity as:

\[
\vec{v} = \vec{v}(x, y, z, t) = u(x, y, z, t)\hat{i} + v(x, y, z, t)\hat{j} + w(x, y, z, t)\hat{k}
\]

where \( \hat{i}, \hat{j}, \) and \( \hat{k} \) are the unit normal vectors in the Cartesian \( x, y, \) and \( z \) directions, respectively.

The *Lagrangian* velocity is the velocity with respect to a point following the flow. E.g., you jump on a parcel of fluid at the hill (say your starting point is \((x_0, y_0, z_0)\)) and record your velocity with respect to a reference frame (in this case a point on the top of the hill, perhaps). We write the Lagrangian velocity as:

\[
\vec{v} = \vec{v}(x_0, y_0, z_0, t) = u(x_0, y_0, z_0, t)\hat{i} + v(x_0, y_0, z_0, t)\hat{j} + w(x_0, y_0, z_0, t)\hat{k}
\]

1.13 This Line and That Line: The Pathline, Streamline, and Streakline

Now that we have described the velocity field we can think of various ways to define lines through the velocity field, each with its own significance.
• **Pathline** – the path traced out by a fluid parcel. E.g., the Lagrangian track of parcels of fluid.

• **Streamline** – A line instantaneously tangent to the velocity field. E.g., if you could freeze the velocity field at an instant in time this is the path a vehicle following the velocity vectors would trace out.

• **Streakline** – the locus of parcels that have previously passed through a given point. E.g., if you rotate a hose from left to right (around a fixed point) the jet of water as it exists in the air is a streakline.

How do we analytically calculate these various lines?

**Pathline**

\[
x = \int u(x_0, y_0, z_0, t) \, dt
\]

\[
y = \int v(x_0, y_0, z_0, t) \, dt
\]

\[
z = \int w(x_0, y_0, z_0, t) \, dt
\]

**Streamline**

\[
\text{Tangent} = \frac{d\vec{r}}{\vec{v}} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}
\]

**Streaklines**

A mathematical challenge!

**1.13.1 Example 1.12 from the textbook**
Chapter 2

Hydrostatics

In many fluid problems the velocity is zero or the velocity is constant \( \Rightarrow \tau = 0 \).

If the tangential stresses are zero this leaves only the normal stresses, which in the absence of acceleration arise only from the pressure. Then our question is one of statics, meaning we have no motion (or relative motion), and is how does the pressure vary? Let’s look at a small fluid element:
Our starting point is \( F = ma = 0 \) hence we write in component form that \( \sum F_x = \sum F_z = 0 \).

\[
\sum F_x = P_x \Delta z \Delta y - P_s \sin \theta \Delta s \Delta y = \frac{\Delta x \Delta z}{2} \Delta y \rho a_x
\]
\[
\sum F_z = P_z \Delta x \Delta y - P_s \cos \theta \Delta s \Delta y - \rho g \frac{\Delta x \Delta z}{2} \Delta y = \frac{\Delta x \Delta z}{2} \Delta y \rho a_z
\]

But from the trigonometry we have

\[
\Delta s \cos \theta = \Delta x; \quad \Delta s \sin \theta = \Delta z
\]

and hence we can write

\[
\sum F_x = P_x - P_s = \rho a_x \frac{\Delta x}{2}
\]
\[
\sum F_z = P_z - P_s = \rho a_z \frac{\Delta z}{2} (g + a_z)
\]

Now, we take the limit as \( \Delta x, \Delta y, \Delta z \to 0 \). Aha!

\[
\sum F_x = P_x - P_s = 0 \implies P_x = P_s
\]
\[
\sum F_z = P_z - P_s = 0 \implies P_z = P_s = P_x
\]

Now, since \( \theta \) is arbitrary this is generically true. In other words the pressure at a point is independent of direction! Hence we speak of pressure as a scalar quantity as it is a quantity with no dependence on direction (as opposed to velocity, which is a vector).

This result is known as Pascal’s law and says that if \( \tau = 0 \) the pressure at a point is the same in all directions.

### 2.1 Equation of Motion in absence of Shear Stresses

Let’s investigate the force balance on a small volume of fluid to determine the equation of motion in the absence of shear stresses.
Surface Forces

\[ d\vec{F_s} = -\nabla P = \frac{\text{Force}}{\text{Volume}} \]

Body Forces

What body forces might act on the fluid element? Gravity, electromagnetic, ...

\[ \Delta \vec{F}_g = \rho g \Delta x \Delta y \Delta z \]

Taking the limit as \( \Delta x, \Delta y, \Delta z \to 0 \) we have

\[ \frac{d\vec{F}_g}{dV} = -\gamma \vec{k} \]
Now, putting it all together we have

\[ \vec{F} = m\vec{a} \]

\[ \frac{d\vec{F}_s}{d\vec{q}} + \frac{d\vec{F}_g}{d\vec{q}} = \rho\vec{a} \]

\[ -\nabla P - \gamma\vec{k} = \rho\vec{a} \]

### 2.2 Hydrostatic Pressure

We found the equation of motion for a fluid with \( \tau = 0 \):

\[ -\nabla P - \gamma\vec{k} = \rho\vec{a} \]

Now, if the fluid is at rest (or at least moving at a constant velocity):

\[ \vec{a} = 0 \implies \nabla P = -\gamma\vec{k} \]

Hence we can write:

\[ \frac{\partial P}{\partial x} = 0, \quad \frac{\partial P}{\partial y} = 0, \quad \frac{\partial P}{\partial z} = -\gamma \]

What does this tell us? Well, we see that \( P = P(z) \) only, hence the pressure at a given elevation (\( z \) position) is constant. In the vertical we have

\[ \frac{dP}{dz} = -\gamma \quad \text{where we have replaced} \ \partial \ \text{with} \ d \ \text{now.} \]

Hence as \( z \uparrow \quad P \downarrow \)